GNSS Data Processing

Theory

Slides

http://www.gage.upc.edu

@ J. Sanz Subirana & J.M. Juan Zornoza
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5 March 2017
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Lecture 6: Differential positioning with code pseudoranges.

Lecture 7: Carrier based differential positioning. Ambiguity resolution techniques.

List of Acronyms.
Introduction

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Master of Science in GNSS
@ J. Sanz & J.M. Juan

Specific Objectives:

- To learn about **GNSS observables** (code and phase), their characteristics, properties, combinations and applications.

- To learn how to calculate **satellites orbits and clocks** from navigation message. To know the achievable precision.

- To learn how to model **pseudodistance** for code and phase measurements. This includes calculation of: 1) Coordinates at emission epoch, 2) Ionospheric delay (Klobuchar model), 3) Tropospheric delay, 4) relativistic correction, 5) clocks offsets and satellite instrumental delays, 6) phase wind-up, etc.

- To learn how to set and solve the navigation equation system using least-squares or Kalman filter (algorithm level).

- To know how to use phase differential positioning: Floating and fixing ambiguities.

- To learn **Carrier Phase Ambiguity Fixing techniques**.

**To get tools and skills to process and analyze GNSS data. To implement algorithms for satellite navigation.**
An intuitive approach to GNSS positioning

A ship determines its location from a set on lighthouses that send an acoustic signal at a known time.

Knowing the emission time “t0” in the lighthouse and the reception time “t1” in the ship, the traveling time “t1-t0”, and the geometric range “ρ = v(t1-t0)” may be computed.

With only one lighthouse there is a whole circumference of possible locations.
A ship determines its location from a set on lighthouses that send an acoustic signal at a known time.

With two lighthouses there are two possible solutions. But, one of them is not on the sea!

With three lighthouses a single solution is found.
Errors in the clocks (lighthouses and ship) synchronism affects the accuracy.

The ranges are measured by means the traveling time of the acoustic signal from the lighthouses to the ship. Thence, the synchronism errors between the lighthouses and ship clocks will degrade the positioning accuracy.

**SUMMARY:** The positioning system is based on:
- To know the coordinates of the lighthouses
- To know the ranges from the ship to the lighthouses
- To solve a geometric problem.

**NOTE:** the ranges are measured by means the traveling time of the acoustic signal from the lighthouses to the ship. Thence, the synchronism errors between the lighthouses and ship clocks will degrade the positioning accuracy.
How GNSS Works

Satellites broadcast orbit and clock data → Satellite coordinates and clock offset

Lighthouses coordinates

Receiver measures traveling time from satellite to receiver → Pseudorange (P)

Lighthouses-ship ranges.

Thence, the receiver coordinates are found solving a geometrical problem: from sat. coordinates and ranges

How GNSS Works

Three measurements puts us at one of two points

One of the solutions is not on the Earth surface.
**How GNSS Works**

**Lesson 1:**
GPS measurements and its combinations

**Measurements: Ranges**

“Pseudoranges” are computed from the traveling time sat-rec

Several error sources affect these measurements

Receiver location

travel time x speed of light
Lesson 2: GPS Orbits and clocks

How GNSS Works

Lesson 3: GPS measurements modeling (code)

Atmospheric propagation: IONO, TROPO

Model:
Atmospheric propagation, relativistic effects, clocks and instrument delays are modeled and removed.
And the navigation equations are built.

How GNSS Works
Lesson 3: Solving the navigation Equations

Navigation equations

The geometric problem is linearized, and Weighted Least Mean Squares or Kalman filter are used to compute the solution.

Model:

Atmospheric propagation, relativistic effects, clocks and instrument delays are modeled and removed.

And the navigation equations are built.

Lessons 4, 5, 6, 7:
Code and Carrier phase Differential positioning. Floating/fixing ambiguities
References


Thank you!
Lecture 1
GNSS measurements and their combinations

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3. Carrier cycle-slips detection.
4. Carrier smoothing of code pseudorange.
GPS SIGNAL STRUCTURE

Two carriers in L-band:
- \( L_1 = 154 \) fo=1575.42 MHz
- \( L_2 = 120 \) fo=1227.60 MHz
where fo=10.23 MHz

- C/A-code for civilian users \([X_C(t)]\)
- P-code only for military and authorized users \([X_P(t)]\)
- Navigation message with satellite ephemeris and clock corrections \([D(t)]\)

GPS SATELLITE SIGNALS

\[
S_{L_1}^{(k)}(t) = a_p X_{P}^{(k)}(t) D^{(k)}(t) \sin(\omega_1 t + \phi_{L_1}) + a_c X_{C}^{(k)}(t) D^{(k)}(t) \cos(\omega_1 t + \phi_{L_1})
\]
\[
S_{L_2}^{(k)}(t) = b_p X_{P}^{(k)}(t) D^{(k)}(t) \sin(\omega_2 t + \phi_{L_2})
\]

GPS Code Pseudorange Measurements

\[
S_{L_1}^{(k)}(t) = a_p X_{P}^{(k)}(t) D^{(k)}(t) \sin(\omega_1 t + \phi_{L_1}) + a_c X_{C}^{(k)}(t) D^{(k)}(t) \cos(\omega_1 t + \phi_{L_1})
\]
\[
S_{L_2}^{(k)}(t) = b_p X_{P}^{(k)}(t) D^{(k)}(t) \sin(\omega_2 t + \phi_{L_2})
\]

\[
P(T) = c \Delta T = c \left[ t_{rec}(T) - t_{Sat}(T - \Delta T) \right]
\]

From hereafter we will call:
- \( C_1 \) pseudorange computed from \( X_C(t) \) binary code (on frequency 1)
- \( P_1 \) pseudorange computed from \( X_P(t) \) binary code (on frequency 1)
- \( P_2 \) pseudorange computed from \( X_P(t) \) binary code (on frequency 2)
GPS Carrier Phase Measurements

\[ S^{(k)}_{i_1}(t) = a_p X^{(k)}_p(t) \, D^{(k)}(t) \sin(\omega t + \varphi_{1t}) + a_c X^{(k)}_c(t) \, D^{(k)}(t) \cos(\omega t + \varphi_{1t}) \]

\[ S^{(k)}_{i_2}(t) = b_p X^{(k)}_p(t) \, D^{(k)}(t) \sin(\omega_2 t + \varphi_{2t}) \]

Carrier phase:

\[ \phi_L(T) = \phi_{L \text{rec}}(T) - \phi_{L \text{sat}}^T (T - \Delta T) \]

\[ = \frac{c}{\lambda} \Delta T + N \]

Unknown ambiguity

From hereafter we will call:

• \( L_1 = \lambda_1 \phi_{L1} \) measur. computed from the carrier phase on frequency 1
• \( L_2 = \lambda_2 \phi_{L2} \) measur. computed from the carrier phase on frequency 2

• \( C_1 \) pseudorange computed from \( X_C(t) \) binary code (on frequency 1)
• \( P_1 \) pseudorange computed from \( X_p(t) \) binary code (on frequency 1)
• \( P_2 \) pseudorange computed from \( X_p(t) \) binary code (on frequency 2)

\[ P_1 \approx \rho + \text{clock offset} \approx 20,000 \text{Km} \]

\( P_1 \) is basically the geometric range (\( \rho \)) between satellite and receiver, plus the relative clock offset.

The range varies in time due to the satellite motion relative to the receiver.
Phase and Code pseudorange measurements

Relative measurement (shifted by the unknown ambiguity \( \lambda N \))

Each time that the receiver lose the phase lock, the unknown ambiguity changes by an integer number of \( \lambda \).

\[
L_1(T) = c \Delta T + \lambda_1 N_1
\]

\( L_1 \approx \rho + \text{clock offset} + \lambda_1 N_1 \)

Code and Carrier Phase measurements

Code (unambiguous but noisier)

Carrier Phase (ambiguous but precise)
GPS measurements: Code and Carrier Phase

Antispoofing (A/S):
The code P is encrypted to Y.
⇒ Only the code C at frequency L1 is available.

<table>
<thead>
<tr>
<th>Code measurements</th>
<th>Wavelength (chip-length)</th>
<th>σ noise (1% of λ) [&quot;]</th>
<th>Main characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>C₁</td>
<td>300 m</td>
<td>3 m</td>
<td>Unambiguous but noisier</td>
</tr>
<tr>
<td>P₁ (Y₁): encrypted</td>
<td>30 m</td>
<td>30 cm</td>
<td></td>
</tr>
<tr>
<td>P₂ (Y₂): encrypted</td>
<td>30 m</td>
<td>30 cm</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Phase measurements</th>
<th>L₁</th>
<th>L₂</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>19.05 cm</td>
<td>24.45 cm</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2 mm</td>
<td>2 mm</td>
<td>Precise but ambiguous</td>
</tr>
</tbody>
</table>

[*] the codes can be smoothed with the phases in order to reduce noise (i.e, C₁ smoothed with L₁ ⇒ 50 cm noise)
GNSS Format Descriptions

- GNSS data files follow a well defined set of standards formats: RINEX, ANTEX, SINEX...
- Understanding a format description is a tough task.
- These standards are explained in a very easy and friendly way through a set of html files.
- Described formats:
  - Observation RINEX
  - Navigation RINEX
  - RINEX CLOCKS
  - SP3 Version C
  - ANTEX

More details at: http://www.gage.upc.edu/glAB

Open GNSS Formats with Firefox internet browser

RINEX measurement file

```
2 OBSERVATION DATA G (GPS)
RGRINEXO V2.4.1 UX AUSLIG 10-JAN-97 10:19
Australian Regional GPS Network (ARGN) - Cocos Island
BIT 2 OF LLI (+4) FLAGS DATA COLLECTED UNDER "AS" CONDITION
-0.000000000103 HARDWARE CALIBRATION (S)
-0.000000000663 CLOCK OFFSET (S)
COC0
Au18
mhr: elelg
126 GUE SNR-8100
327 DORNE MARGOLIN T
-74.19503244 61.90691.5624 -13377.9813
0.0040 0.0000 0.0000
1 1
5 C1 L1 L2 P1 0
SNR is mapped to signal strength [0,1,4-9]
SNR: >500 >100 >50 >10 >5 >0 bad n/a
9 8 7 6 5 4 1 0
1997 1 9 0 7 30.0000000
1997 1 9 23 59 30.0000000
```

More details at: http://www.gage.upc.edu/glAB
### RINEX measurement file

**Observation Data**

<table>
<thead>
<tr>
<th>G (GPS)</th>
<th>Epoch Flag</th>
<th>Number of Tracked Satellites</th>
<th>Time of First Obs</th>
<th>Time of Last Obs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1997</td>
<td>1</td>
<td>9</td>
<td>0</td>
<td>7</td>
</tr>
</tbody>
</table>

**Measurement Time**

- Number of tracked satellites: 9
- Time of first obs: 1997, 1, 9, 0, 7
- Time of last obs: 1997, 1, 9, 23, 69
- Number of observations: 30
- One satellite per row

**APPEND POSITION XYZ**

<table>
<thead>
<tr>
<th>Snr:</th>
<th>&gt;500</th>
<th>&gt;100</th>
<th>&gt;5</th>
<th>&gt;10</th>
<th>bad</th>
<th>n/a</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
</tr>
</tbody>
</table>

**WAVELLENGTH FACT L1/2**

<table>
<thead>
<tr>
<th>#</th>
<th>TYPES OF OBSERV</th>
<th>COMMENT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

**Antenna: Delta H/E/N**

- Time of first obs: 1997, 1, 9, 0, 7
- Time of last obs: 1997, 1, 9, 23, 69
- Number of observations: 30
- One satellite per row

**Time of First Obs:**

1997, 1, 9, 0, 7

**Time of Last Obs:**

1997, 1, 9, 23, 69

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RI NEX measurement file

<table>
<thead>
<tr>
<th>OBSERVATION DATA</th>
<th>RINEX VERSION / TYPE</th>
</tr>
</thead>
<tbody>
<tr>
<td>COOS ISLAND</td>
<td>10-JAN-97 10:19</td>
</tr>
</tbody>
</table>

**Flagged Data Collected Under "AS" Condition**

<table>
<thead>
<tr>
<th>C1</th>
<th>L1</th>
<th>L2</th>
<th>P2</th>
<th>P1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

**SNR Indicators**

- SNR is mapped to signal strength [0, 1, 4-9]
- CM: 900 >100 >50 >30 >20 >10 >6 >4 >2 >1
- BAD: n/a

**Tropospheric Delay**

$$P_{sat}^{rec} = \rho_{rec}^{sat} + c \cdot (dt_{rec} - dt_{sat}) + \sum \delta$$

- Geometric range
- Clock offsets
- Tropospheric delay
- Ionospheric delay
- Instrumental delays
- Noise

$\sum \delta = Trop^{sat}_{rec} + Ion^{sat}_{rec} + K_{rec} + K^{sat} + \epsilon$
Exercise:

a) Using the file coco0090.97o, generate the “txt” file coco0090.97.txt (with data ordered in columns).

b) Plot code and phase measurements for satellite PRN28 and discuss the results.

Resolution:

a) gLAB_linux -input:cfg meas.cfg -input:obs coco0090.97o

b) See next plots:

The RINEX file is converted to a “columnar format” to easily plot its contents and to analyze the measurements (the public domain free tool “gnuplot” is used in the book to make the plots).
The geometry “$\rho$” is the dominant term in the plot. The pattern in the figures is due to the variation of “$\rho$”.

\[ P_{1\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + c \cdot (dt_{\text{sta}}^{\text{sat}} - dt_{\text{sta}}^{\text{sat}}) + Trop_{\text{sta}}^{\text{sat}} + Ion_{\text{1sta}}^{\text{sat}} + K_{\text{1sta}}^{\text{sat}} + K_{\text{1}}^{\text{sat}} + \varepsilon_{1} \]

Similar plot for code measurements at $f_2$.

Notice that

- Ionosphere (Ion) and
- Instrumental delays (K)

depend on frequency.

\[ P_{2\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + c \cdot (dt_{\text{sta}}^{\text{sat}} - dt_{\text{sta}}^{\text{sat}}) + Trop_{\text{sta}}^{\text{sat}} + Ion_{\text{2sta}}^{\text{sat}} + K_{\text{2sta}}^{\text{sat}} + K_{\text{2}}^{\text{sat}} + \varepsilon_{2} \]
Ionosphere delays code and advances phase measurements

**Code measurements: C1,P1,P2**

\[ P_{1\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + c \cdot (dt_{\text{sta}} - dt_{\text{sat}}) + T\text{rop}_{\text{sta}}^{\text{sat}} + \text{Ion}_{1\text{sta}}^{\text{sat}} + K_{1\text{sta}} + K_{1} + \varepsilon_{1} \]

\[ P_{2\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + c \cdot (dt_{\text{sta}} - dt_{\text{sat}}) + T\text{rop}_{\text{sta}}^{\text{sat}} + \text{Ion}_{2\text{sta}}^{\text{sat}} + K_{2\text{sta}} + K_{2} + \varepsilon_{2} \]

**Phase measurements: L1,L2**

\[ L_{1\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + c \cdot (dt_{\text{sta}} - dt_{\text{sat}}) + T\text{rop}_{\text{sta}}^{\text{sat}} - \text{Ion}_{1\text{sta}}^{\text{sat}} + b_{1\text{sta}} + b_{1} + \lambda_{1}N_{1} + \lambda_{w} + v_{1} \]

\[ L_{2\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + c \cdot (dt_{\text{sta}} - dt_{\text{sat}}) + T\text{rop}_{\text{sta}}^{\text{sat}} - \text{Ion}_{2\text{sta}}^{\text{sat}} + b_{2\text{sta}} + b_{2} + \lambda_{2}N_{2} + \lambda_{w} + v_{2} \]

The geometry \( \rho \) is the dominant term in the plot. The pattern in the figures is due to the variation of \( \rho \).

The curves are broken when the receiver loss the lock (cycle-slip).

When a cycle-slip happens, the phase measurement \( L \) changes by an unknown integer number of cycles \( N \).
When a cycle-slip happens, the phase measurement "$L$" changes by an unknown integer number of cycles ($N$).

$$L_{\text{2sta}}^{\text{sat}} = p_{\text{sta}}^{\text{sat}} + c \cdot (dt_{\text{sta}} - dt_{\text{sat}}) + Trop_{\text{sta}}^{\text{sat}} - Ion_{\text{2sta}}^{\text{sat}} + b_{\text{2sta}} + b_{\text{sat}} + \lambda_2 N_2 + \lambda_2 w + \nu_2$$

The geometry "$\rho$" is the dominant term in the plot. The pattern in the figures is due to the variation of "$\rho$".

The curves are broken when the receiver loses the lock (cycle-slip).

Contents

1. Review of GNSS measurements.
2. Linear combinations of measurements.
3. Carrier cycle-slips detection.
4. Carrier smoothing of code pseudorange.
Linear Combinations of measurements:

- Geometry-free (or Ionospheric) combination.
- Ionosphere-Free combination.
- Wide-lane and Narrow-lane combinations.

1. Geometry-free (or Ionospheric) combination

\[ P_1 = P_2 - P_1 = \text{Iono} + \text{ctt} \]

\[ L_1 = L_1 - L_2 = \text{Iono} + \text{ctt} + \text{Ambig} \]

**Code measurements:** \( C_1, P_1, P_2 \)

\[
P_{1,\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + c \cdot (d_{\text{sta}} - d_{\text{sat}}) + Trop_{\text{sta}}^{\text{sat}} + \text{Ion}^{\text{sat}}_{1,\text{sta}} + K_{1,\text{sta}}^{\text{sat}} + K_{1}^{\text{sat}} + \varepsilon_1
\]

\[
P_{2,\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + c \cdot (d_{\text{sta}} - d_{\text{sat}}) + Trop_{\text{sta}}^{\text{sat}} + \text{Ion}^{\text{sat}}_{2,\text{sta}} + K_{2,\text{sta}}^{\text{sat}} + K_{2}^{\text{sat}} + \varepsilon_2
\]

**Carrier measurements:** \( L_1, L_2 \)

\[
L_{1,\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + c \cdot (d_{\text{sta}} - d_{\text{sat}}) + Trop_{\text{sta}}^{\text{sat}} + \text{Ion}^{\text{sat}}_{1,\text{sta}} + b_{1,\text{sta}}^{\text{sat}} + \lambda_1 N_1 + \lambda_1 w + v_1
\]

\[
L_{2,\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + c \cdot (d_{\text{sta}} - d_{\text{sat}}) + Trop_{\text{sta}}^{\text{sat}} + \text{Ion}^{\text{sat}}_{2,\text{sta}} + b_{2,\text{sta}}^{\text{sat}} + \lambda_2 N_2 + \lambda_2 w + v_2
\]
1. Geometry-free (or ionospheric) combination

\[ P_1 = P_2 - P_1 = \text{Iono} + ctt \]

\[ L_1 = L_1 - L_2 = \text{Iono} + ctt + \text{Ambig} \]

- The pattern corresponds to the ionospheric refraction (Ion), because the other terms (K) are constant.
- Notice that code measurements are noisier.

**Ionospheric effects**

The ionospheric refraction depends on:
- Geographic location
- Time of day
- Time with respect to solar cycle (11y)
2. Ionosphere-free Combination \((P_c, L_c)\)

The ionospheric refraction depends on the inverse of the squared frequency and can be removed up to 99.9% combining \(f1\) and \(f2\) signals:

\[
P_c = \frac{f_1^2 P_1 - f_2^2 P_2}{f_1^2 - f_2^2}
\]

\[
L_c = \frac{f_1^2 L_1 - f_2^2 L_2}{f_1^2 - f_2^2}
\]

Note: \(K_{sat}\) cancels in \(P_c\) and \(K_{sta}\) included in \(dt_{sta}\)

- \(Pc_{sat} = \rho_{sat} + c \cdot (dt_{sta} - dt_{sat}) + Trop_{sat} + \varepsilon_c\)
- \(Lc_{sat} = \rho_{sat} + c \cdot (dt_{sta} - dt_{sat}) + Trop_{sat} + \lambda_N W_{sat} + b_{c,sta} + b_{c,sat} + \lambda_N (N_{sat} - \lambda_N N_{sat}) + N_{wc}\)

\[\lambda_N = 10.7 \text{ cm}, \lambda_W = 86.2 \text{ cm}\]

The ionospheric refraction has been removed in \(L_c\) and \(P_c\)

\[N_W = N_1 - N_2\]
Comments:

Two-frequency receivers are needed to apply the ionosphere-free combination.

If a one-frequency receiver is used, a ionospheric model must be applied to remove the ionospheric refraction. The GPS navigation message provides the parameters of the Klobuchar model which accounts for more than 50% (RMS) of the ionospheric delay.

3.- Narrow-lane ($P_N$) and Wide-lane Combination ($L_W$)

The wide-lane combination $L_W$ provides a signal with a large wavelength ($\lambda_W=86.2\text{cm} \sim 4\times\lambda_I$). This makes it very useful for detecting cycle-slips through the Melbourne-Wübbena combination: $L_W - P_N$

$$P_N = \frac{f_1 P_1 + f_2 P_2}{f_1 + f_2} \quad L_W = \frac{f_1 L_1 - f_2 L_2}{f_1 - f_2}$$

The same sign

The ambiguities $N_W$ are INTEGER Numbers!

$N_W = N_1 - N_2$
1) Consider the wide-lane combination of carrier phase measurements

\[ L_W = \frac{f_1 L_1 - f_2 L_2}{f_1 - f_2} \]

where \( L_W \) is given in length units (i.e. \( L_i = \lambda_i \phi_i \)).

Show that the corresponding wavelength is:

\[ \lambda_W = \frac{c}{f_1 - f_2} \]

**Hint:**

\[ L_W = \lambda_w \phi_W ; \quad \phi_W = \phi_1 - \phi_2 \]

2) Assuming \( L_1, L_2 \) uncorrelated measurements with equal noise \( \sigma_L \), show that:

\[ \sigma_{L_W} = \frac{\gamma_{12} + 1}{\sqrt{\gamma_{12} - 1}} \sigma_L ; \quad \gamma_{12} = \left( \frac{f_1}{f_2} \right)^2 \]

---

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Detecting cycle-slips

This cycle-slip involves millions of cycles ➔ it is easy to detect!!

There is a cycle-slip of only one cycle (~20cm) ➔ How to detect it?

Exercise:

a) Using the file 95oct18casa___r0.rnx, generate the “txt” file 95oct18casa.a (with data ordered in columns).

b) Insert a cycle-slip of “one wavelength” (19cm) in L1 measurement at t=5000 s (and no cycle-slip in L2).

c) Plot the measurements “L1, L1-P1, LC-PC, Lw-PN and L1-L2” and discuss which combination/s should be used to detect the cycle-slip.

Resolution:

a) gLAB_linux -input:cfg meas.cfg -input:obs 95oct18casa_r0.rnx

b) cat 95oct18casa.a | gawk '{if ($4==18) print $3,$5,$6,$7,$8}' > s18.org
cat s18.org | gawk '{if ($1>=5000) $2=$2+0.19; printf “%s %f %f %f %f \n”, $1,$2,$3,$4,$5}' > s18.cl

c) See next plots:
The geometry “\( \rho \)” is the dominant term in the plot. The variation of “\( \rho \)” in 1 sec may be hundreds of meters, many times greater than the cycle-slip (19 cm) → the variation of \( \rho \) shadows the cycle-slip!

\[
L_{1,\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + c \cdot (d_{\text{sta}}^{\text{sat}} - d_{\text{sat}}^{\text{sta}}) + \text{Trop}_{\text{sta}}^{\text{sat}} - \text{Ion}_{1,\text{sta}}^{\text{sat}} + b_{1,\text{sta}}^{\text{sat}} + b_{1}^{\text{sat}} + \lambda_{1} N_{1,\text{sta}}^{\text{sat}} + \lambda_{1} w_{\text{sta}}^{\text{sat}} + v_{1}
\]

The geometry and clock offsets have been removed. The trend is due to the Ionosphere. The P1 code noise shadows the cycle-slip, and without the reference (in blue), the time where the cycle-slip happens could not be identified.

A jump of \( \lambda = 19 \, \text{cm} \) (one cycle in L1) has been introduced in L1 at \( t=5000s \)
The geometry and clock offsets have been removed.
The trend is due to the Ionosphere. The \( P1 \) code noise shadows the cycle-slip, and without the reference (in blue), the time where the cycle-slip happens could not be identified.

\[
P_{1,\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + e \cdot (dt_{\text{sta}} - dt_{\text{sat}}) + T_{\text{sta}}^{\text{sat}} + I_{\text{sta}}^{\text{sat}} + K_{1,\text{sta}} + K_{1}^{\text{sat}} + \varepsilon_{1}
\]

\[
L_{1,\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + e \cdot (dt_{\text{sta}} - dt_{\text{sat}}) + T_{\text{sta}}^{\text{sat}} - I_{\text{sta}}^{\text{sat}} + b_{4,\text{sta}} + b_{1}^{\text{sat}} + \lambda_{N}^{\text{sat}}_{\text{sta}} + \lambda_{I}^{\text{sat}} + \lambda_{W}^{\text{sat}}_{\text{sta}} + \nu_{1}
\]

A jump of \( \lambda=19 \text{ cm} \) (one cycle in \( L1 \)) has been introduced in \( L1 \) at \( t=5000s \)

The geometry, clock offsets and iono have been removed.
There is a constant pattern plus noise. The \( P_c \) code noise also shadows the cycle-slip, and without the reference (in blue), the time where the cycle-slip happens could not be identified.

\[
P_{c,\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + e \cdot (dt_{\text{sta}} - dt_{\text{sat}}) + T_{\text{sta}}^{\text{sat}} + \varepsilon_{c}
\]

\[
L_{c,\text{sta}}^{\text{sat}} = \rho_{\text{sta}}^{\text{sat}} + e \cdot (dt_{\text{sta}} - dt_{\text{sat}}) + T_{\text{sta}}^{\text{sat}} + \lambda_{N}^{\text{sat}}_{\text{sta}} + b_{c,\text{sta}} + b_{c}^{\text{sat}} + \lambda_{N}^{\text{sat}}_{\text{sta}} - \lambda_{N}^{\text{sat}}_{\text{sta}} + \nu_{c}
\]

A jump of \( \lambda=19 \text{ cm} \) (one cycle in \( L1 \)) has been introduced in \( L1 \) at \( t=5000s \)
The geometry, clock offsets and iono have been removed. There is a constant pattern plus noise. The $P_l$ code noise also shadows the cycle-slip, and without the reference (in blue), the time where the cycle-slip happens could not be identified.

\[ L_{1,\text{sat}} - P_{1,\text{sat}} = ctt + \text{ambig} + \epsilon \]

A jump of $\lambda = 19$ cm (one cycle in $L_1$) has been introduced in $L_1$ at $t = 5000s$.

The geometry, clock offsets and iono have been removed. There is a constant pattern plus noise. The $P_N$ code noise is under one cycle of $L_w$. Thence, the cycle-slip is clearly detected.

\[ L_{w,\text{sat}} - P_{w,\text{sat}} = ctt + \text{ambig} + \epsilon \]

A jump of $\lambda = 19$ cm (one cycle in $L_1$) has been introduced in $L_1$ at $t = 5000s$. 

1 unit = 5.4 cm 
1 unit = 86.2 cm (Lw cycles)
The geometry and clock offsets have been removed. The trend is due to the Iono. The $L_1$ carrier noise is few mm, and the variation of the ionosphere in 1 second is lower than $\lambda_1 = 19 \text{ cm}$. Thence, the cycle-slip is detected.

A jump of $\lambda=19 \text{ cm}$ (one cycle in $L_1$) has been introduced in $L_1$ at $t=5000s$. The geometry and clock offsets have been removed. The trend is due to the Iono. The $L_1$ carrier noise is few mm, and the variation of the ionosphere in 1 second is lower than $\lambda_1 = 19 \text{ cm}$. Thence, the cycle-slip is detected.

**Summary**

- $L_1$
- $L_1$-PI
- LC-PC
The cycle-slips are detected by the Ionospheric combination (LI=L1-L2) and the Melbourne Wübbena (MW=Lw-PN)

Two independent combinations, LI and Lw, allow to detect two independent cycle-slips (in L1 and L2 phase measur.).

Notice that, from L1, L2 is not possible to detect small cycle-slips

Summary of Cycle-slip detectors

<table>
<thead>
<tr>
<th>COMBINATION</th>
<th>MEAS</th>
<th>Combination Noise (σ)</th>
<th>λ</th>
<th>σ/λ</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1-P1</td>
<td>-2·Ion+K+λ1·N1</td>
<td>σL1-P1 ≈ σp1 = 30 cm</td>
<td>λ1 = 19.0 cm</td>
<td>1.58</td>
</tr>
<tr>
<td>Lc-Pc</td>
<td>k_c+λN·Rc</td>
<td>σLc-Pc ≈ σpC = 2.98 σp = 89 cm</td>
<td>λN = 10.7 cm</td>
<td>8.32</td>
</tr>
<tr>
<td>Lr-P1</td>
<td>k1+λ1·N1-λ2·N2</td>
<td>σLr-P1 ≈ σp = 42 cm</td>
<td>λ2-λ1 = 5.4 cm</td>
<td>7.78</td>
</tr>
<tr>
<td>Lw-PN</td>
<td>λw·Nw</td>
<td>σLw-PN ≈ σpN = σp/√2 = 21 cm</td>
<td>λw = 86.2 cm</td>
<td>0.25</td>
</tr>
<tr>
<td>L1</td>
<td>Ion+k1+λ1·N1-λ2·N2</td>
<td>σL1 = √2 σL1 = 3 mm</td>
<td>λ2-λ1 = 5.4 cm</td>
<td>0.06</td>
</tr>
</tbody>
</table>
Contents

1. Review of GNSS measurements.
2. Linear combinations of measurements.
3. Carrier cycle-slips detection.
   3.1 Cycle-slip Detection Algorithms
4. Carrier smoothing of code pseudorange.

Cycle-slip detector based on carrier phase data:
The Geometry-free combination

Input data: Geometry-free combination of carrier phase measurements

\[ L_1 = L_1(s,k) - L_2(s,k) \]

Output: [satellite, time, cycle-slip flag].

For each epoch \(k\)

For each tracked satellite \((s)\)

- Declare cycle slip when data gap greater than \(t_{\Delta t} \).
- Fit a second-degree polynomial \(p(s;k)\) to the previous values (after the last cycle-slip) \([L_1(s,k-N_1),\ldots,L_1(s,k-1)]\).
- Compare the measured \(L_1(s;k)\) and the predicted value \(p(s;k)\) at epoch \(k\). If the discrepancy exceeds a given \(\text{threshold}\), then declare cycle slip. That is, if \(|L_1(s;k) - p(s;k)| > \text{threshold}\) then cycle slip.
- Reset algorithm after cycle slip.

End
End

The detection is based on fitting a second order polynomial over a sliding window of \(N_t\) samples.
The predicted value is compared with the observed one to detect cycle-slip.
Under not disturbed ionospheric conditions, the geometry-free combination performs as a very precise and smooth test signal, driven by the ionospheric refraction. Although, for instance, the jump produced by a simultaneous one-cycle slip in both signals is smaller in this combination than in the original signals ($\Delta_2-\Delta_1 = 5.4\text{cm}$), it can provide reliable detection even for small jumps.

Cycle-slip detector based on code and carrier phase data: The Melbourne-Wübbena combination

**Input data:** Melbourne-Wübbena combination

$$B_W = L_W - P_N = \lambda W N_W + \epsilon$$

**Output:** [satellite (PRN), time, cycle-slip flag]

For each epoch ($k$)

For each tracked satellite ($s$)

- Declare cycle-slip when data hole greater than $tol_{\Delta t}$ (e.g., 60 s).
- If no data hole larger than $tol_{\Delta t}$, thence:
  - Compare the measurement $B_W(s; k)$ at the epoch $k$ with the mean bias $m_{BW}(s; k - 1)$ computed from the previous values. If the discrepancy is over a threshold $= K_{factor} \times S_{BW}$ (e.g., $K_{factor} = 4$), declare cycle-slip. That is:
    $$\text{If } |B_W(s; k) - m_{BW}(s; k - 1)| > K_{factor} S_{BW}(s; k - 1),$$
    thence, cycle-slip.
- Update the mean and sigma values according to the equations:
  $$m_{BW}(s; k) = \frac{k-1}{k} m_{BW}(s; k - 1) + \frac{1}{k} B_W(s; k)$$
  $$S_{BW}^2(s; k) = \frac{k-1}{k} S_{BW}^2(s; k - 1) + \frac{1}{k} (B_W(s; k) - m_{BW}(s; k - 1))^2$$

Note the $S_{BW}$ is initialised with an a priori $S_0 = \lambda W / 2$.
Nevertheless, in spite of these benefits, the performance is worse than in the previous carrier-phase-only based detector and it is used as a secondary test.

**Exercises:**

1) Show that  $\Delta N_1 = 9$ and $\Delta N_2 = 7$ produces jumps of few millimetres in the geometry-free combination.

2) Show that no jump happens in the geometry-free combination when  $\Delta N_1 / \Delta N_2 = 77 / 60$. In particular when $\Delta N_1=77$ and $\Delta N_2=60$ the jump in the wide-lane combination is: $17\lambda_W = 15m$

**Hint:** Consider the following relationships (from [RD-1]):

The effect of a jump in the integer ambiguities in terms of $\Delta N_1$, $\Delta N_2$ and $\lambda_W$ is given next:

\[
\Delta \phi_w, \Delta \phi_f, \Delta \phi_c \text{ variations}
\]

\[
\Delta \phi_w = \lambda_w \Delta N_w = \lambda_w (\Delta N_1 - \Delta N_2) \\
\Delta \phi_f = \lambda_1 \Delta N_1 - \lambda_2 \Delta N_2 = (\lambda_2 - \lambda_1) \Delta N_1 + \lambda_0 \Delta N_w \\
\Delta \phi_c = \lambda_N \left( \frac{\lambda_w}{\lambda_1} \Delta N_1 - \frac{\lambda_w}{\lambda_2} \Delta N_2 \right) = \lambda_N \left( \Delta N_1 + \frac{\lambda_w}{\lambda_2} \Delta N_w \right)
\]
Example of Single frequency Cycle-slip detector

Input data: Code pseudorange ($P$) and carrier phase ($L$) measurements.

Output: [satellite (PRN), time, cycle-slip flag]

For each epoch ($k$)

For each tracked satellite ($s$)

- Declare cycle-slip when data hole greater than $tol_{\Delta r}^{24}$.
- If no data hole larger than $tol_{\Delta r}$, thence:
- Update an array with the last $N$ differences of

$$d(s; k) = L_1 (s; k) - P_1 (s; k)$$

That is: $[d(s; k - N), \ldots, d(s; k - 1)]$

- Compute the mean and sigma discrepancy over the previous $N$ samples [$k - N, \ldots, k - 1$]:

$$m_d (s; k - 1) = \frac{1}{N} \sum_{i=1}^{N} d(s; k - i)$$

$$m_{d^2} (s; k - 1) = \frac{1}{N} \sum_{i=1}^{N} d^2(s; k - i)$$

$$S_d (s; k - 1) = \sqrt{m_{d^2} (s; k - 1) - m_d^2 (s; k - 1)}$$

- Compare the difference at the epoch $k$ with the mean value of differences computed over the previous $N$ samples window. If the value is over a threshold $\sigma_d = S_d$ (e.g., $\sigma_d = 5$), declare cycle-slip.

That is:

If $|d(s; k) - m_d (s; k - 1)| > n_\sigma S_d (s; k - 1)$,

Thence, cycle-slip.

The detection is based on real-time computation of mean and sigma values of the differences ($d=L_1-P_1$) of the code pseudorange and carrier over a sliding window of $N$ samples (e.g. $N=100$).

A cycle-slip is declared when a measurement differs from the mean bias value over a predefined threshold.

This detector is affected by the code pseudorange noise and multipath as well as the divergence of the ionosphere.

Higher sampling rate improves detection performance, but shortest jumps can still escape from this detector.

On the other hand, a minimum number of samples is needed for filter initialization in order to ensure a reliable value of sigma for the detection threshold.

More details, exercises and examples of software code implementation of these detectors can be found in [RD-1] and [RD-2].
Carrier smoothing of code pseudorange

The noisy (but unambiguous) code pseudorange can be smoothed with the precise (but ambiguous) carrier. A simple algorithm is given next:

**Hatch filter:**

\[
\hat{P}(k) = \frac{1}{n} P(k) + \frac{n-1}{n} \left( \hat{P}(k-1) + L(k) - L(k-1) \right)
\]

where \( \hat{P}(1) = P(1) \) and

\[n = k; \quad k < N\]
\[n = N; \quad k \geq N\]

This algorithm can be interpreted as a real-time alignment of the carrier phase with the code measurement:

\[
\hat{P}(k) = \frac{1}{n} P(k) + \frac{n-1}{n} (\hat{P}(k-1) + L(k) - L(k-1)) = L(k) + \langle P - L \rangle_{(k)}
\]
This algorithm can be interpreted as a real-time alignment of the carrier phase with the code measurement:

\[ \hat{P}(k) = \frac{1}{n} P(k) + \frac{n-1}{n} \left( \hat{P}(k-1) + L(k) - L(k-1) \right) = L(k) + \langle P - L \rangle_{(k)} \]
Time varying ionosphere induces a bias in the single frequency (SF) smoothed code when it is averaged in the smoothing filter (Hatch filter).

Let: 

\[ P_1 = \rho + I_1 + \varepsilon_1 \]

\[ L_1 = \rho - I_1 + B_1 + \zeta_1 \]

thence,

\[ P_1 - L_1 = 2I_1 - B_1 + \varepsilon_1 \Rightarrow 2I_1 : \text{Code-carrier divergence} \]

Substituting \( P_1 - L_1 \) in Hatch filter equation

\[ \hat{P}(k) = L(k) + \langle P - L \rangle_{(k)} = \rho(k) - I_1(k) + B_1 + \langle 2I_1 - B_1 \rangle_{(k)} = \]

\[ = \rho(k) + I_1(k) + 2\langle I_1 \rangle_{(k)} - I_1(k) \]

\[ \Rightarrow \hat{P}_1 = \rho + I_1 + \text{bias}_I + \nu_1 \]

where, being the ambiguity term \( B_1 \) a constant bias, thence \( \langle B_1 \rangle_{(k)} \approx B_1 \), and cancels in the previous expression.

where \( \nu_1 \) is the noise term after smoothing.
Iono

PRN03: C1 3600s smoothing and divergence of ionosphere

- C1 Raw
- C1 SF smoothed (3600)
- C1 DFree smoothed (3600)

N=3600 s
Halloween storm

Data File: amc23030.03o_1Hz

$N=100$ (i.e. filter smoothing time constant $\tau=100$ sec).

[+ \text{Ionospheric Threat Parameterization for Local Area Global-Positioning-System-Based Aircraft Landing Systems, Datta-Barua et al, Journal of Aircraft Vol. 47, No. 4, July-August 2010, DOI: 10.2514/1.46719}]

Fig. 2 Map of equivalent vertical ionospheric delay over eastern United States on 20 Nov. 2003 at 20:15 UT.
Divergence-Free (Dfree) smoother:

With two frequency carrier measurements a combination of carriers with the same ionospheric delay (the same sign) as the code can be generated:

\[
L_{1,DF} = L_1 + 2\tilde{\alpha}_1(L_1 - L_2) = \rho + I_1 + B_{1,DF} + \zeta_{1,DF}
\]

\[
\tilde{\alpha}_1 = \frac{f_1^2}{f_1^2 - f_2^2} = \frac{1}{\gamma - 1} = 1.545
\]

\[
\gamma = \left(\frac{77}{60}\right)^2
\]

With this new combination we have:

\[
P_1 = \rho + I_1 + \varepsilon_1
\]

\[
L_{1,DF} = \rho + I_1 + B_{1,DF} + \zeta_{1,DF}
\]

Thence,

\[
P_1 - L_{1,DF} = B_{1,DF} + \varepsilon_1
\]

\[
\Rightarrow \text{No Code-carrier divergence!}
\]

\[
\hat{P}_{1,DF} = \rho + I_1 + \nu_{12}
\]

This smoothed code is immune to temporal gradients (unlike the SF smoother), being the same ionospheric delay as in the original raw code (i.e. \(I_1\)). Nevertheless, as it is still affected by the ionosphere, its spatial decorrelation must be taken into account in differential positioning.
Ionosphere-Free (Ifree) smoother: Using both code and carrier dual-frequency measurements, it is possible to remove the frequency dependent effects using the ionosphere-free combination of code and carriers (PC and LC). Thence:

\[
P_C = \rho + \varepsilon_{PC} \\
L_C = \rho + B_{LC} + \nu_{LC}
\]

\[
P_{\text{Ifree}} \equiv P_C = \frac{\gamma P_1 - P_2}{\gamma - 1} ; \quad L_{\text{Ifree}} \equiv L_C = \frac{\gamma L_1 - L_2}{\gamma - 1}, \quad \gamma = \left(\frac{77}{60}\right)^2
\]

Thence,

\[
P_C - L_C = B_{C} + \varepsilon_{P_C} \implies \hat{P}_{\text{Ifree}} \equiv \hat{P}_C = \rho + \nu_{\text{Ifree}}
\]

This smoothed is based on the ionosphere-free combination of measurements, and therefore it is unaffected by either the spatial and temporal inospheric gradients, but has the disadvantage that the noise is amplified by a factor 3 (using the legacy GPS signals).
Exercise:

Justify that the ionosphere-free combination (PC) is (obviously) not affected by the code-carrier divergence, but it is 3 times noisier.
Halloween storm

Data File: amc23030.03o_1Hz

$\text{STEC PRN13 (shifted)}$

$\text{STEC}$

$N=100$ (i.e. filter smoothing time constant $\tau=100 \text{ sec}$).

@ J. Sanz & J.M. Juan
## Contents

1. Review of GNSS measurements.
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### Multipath

One or more reflected signals reach the antenna in addition to the direct signal. Reflective objects can be earth surface (ground and water), buildings, trees, hills, etc.

It affects both code and carrier phase measurements, and it is more important at low elevation angles.

- **Code:** up to 1.5 chip-length ➔ up to 450m for C1 [theoretically]
  - Typically: less than 2-3 m.

- **Phase:** up to $\lambda/4$ ➔ up to 5 cm for L1 and L2 [theoretically]
  - Typically: less than 1 cm
Exercise

Plot code and phase geometry-free combination for satellite PRN 15 of file 97jan09coco_r0.rnx and discuss the results.

Butterfly shape:
High multipath for low elevation rays (when satellite rises and sets)
PC code multipath: upc3: PRN20

\[ M_{Pc} = P_c - L_c \]

Melbourne-Wubbena multipath: hv1: PRN03

\[ M_{MW} = P_N - L_W \]
After one year, the directions of the Sun and Aries coincide again, but the number of laps relative to the Sun (solar days) is one less than those relative to Aries (sidereal days).

\[
\frac{24h}{365.2422} = 3^m 56^s
\]

Thus, a **sidereal day is shorter** than a solar day for about **3\(^m\) 56\(^s\)**
GPS standalone (C1 code)  

**Receiver noise and multipath**

Barcelona, Spain: 2005-6-29  
Receiver: NovAtel OEM3, Antenna: NovAtel 800 (Pinwheel)

---

**Same environment!**

GPS standalone (C1 code)  

**Receiver noise and multipath**

Barcelona, Spain: 2005-6-29  
Receiver: Trimble Lassen SK-II, Antenna: Compact Magnetic-Mount

---

10,000 €

100 €
References


Lecture 2

Satellite orbits and clocks computation and accuracy

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5 March 2017
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4. GPS Satellite clock computation and accuracy
   4.1. From Broadcast Navigation Message.
   4.2. From precise products.
5. Geographic decorrelation of ephemeris errors.
The GPS navigation message provides pseudo-Keplerian elements to compute satellite coordinates.

6 values are needed \((x,y,z,vx,vy,vz)\) to provide the position and velocity of a body. They can be map into the six Keplerian elements \((a, e, i, \Omega, \omega, V)\), which provides the "natural" representation of the orbit!
(a, e, i, Ω, ω, V)

orbit shape
orbit orientation
position in the orbit

i  inclination
ω  argument of perigee
Ω  arg. ascending node (Aries)
λ  arg. ascending node (Greenwich)
V  true anomaly
θ  sidereal time
γ  vernal equinox
G  Greenwich meridian

Master of Science in GNSS
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Due to the non-spherical nature of gravitational potential, the attraction of the Sun and Moon, the solar radiation pressure, etc., the true satellite path deviates from the elliptic orbit.

At any time an elliptical orbit tangent to the true path can be defined. This is the “osculating orbit”, whose Keplerian elements vary with time “t”:

\[ a(t), e(t), i(t), \Omega(t), \omega(t), V(t) \]
Different magnitudes of perturbation and their effects on GPS orbits

<table>
<thead>
<tr>
<th>Perturbation</th>
<th>Acceleration (m/s²)</th>
<th>Orbital effect (in 3 hours)</th>
<th>Orbital effect (in 3 days)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central force (as a reference)</td>
<td>0.56</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$J_2$</td>
<td>$5 \times 10^{-5}$</td>
<td>2 km</td>
<td>14 km</td>
</tr>
<tr>
<td>Rest of the harmonics</td>
<td>$3 \times 10^{-7}$</td>
<td>50–80 m</td>
<td>100–1500 m</td>
</tr>
<tr>
<td>Solar + Moon grav.</td>
<td>$5 \times 10^{-6}$</td>
<td>5–150 m</td>
<td>1000–3000 m</td>
</tr>
<tr>
<td>Tidal effects</td>
<td>$1 \times 10^{-9}$</td>
<td>–</td>
<td>0.5–1.0 m</td>
</tr>
<tr>
<td>Solar rad. pressure</td>
<td>$1 \times 10^{-7}$</td>
<td>5–10 m</td>
<td>100–800 m</td>
</tr>
</tbody>
</table>

GLONASS Broadcast orbit integration terms
Session 3.1, Ex8c: Inclination

- Central body
- Central body + J2
- Central body + J2 + SM

Session 3.1, Ex8c: Argument of Perigee

- Central body
- Central body + J2
- Central body + J2 + SM
Calculation of osculating orbital elements from position and velocity (\textit{rv2osc.f})

\[
\begin{align*}
(x, y, z, v_x, v_y, v_z) & \Rightarrow (a, e, i, \Omega, \omega, M) \\
\vec{c} = \vec{r} \times \vec{v} & \Rightarrow p = c^2 \mu \Rightarrow p \\
v^2 = \mu(2/r - 1/a) & \Rightarrow a \\
p = a \left(1 - e^2\right) & \Rightarrow e \\
\vec{c} = \vec{c}^2 & \Rightarrow \Omega = \arctan(-c_x/c_y); \quad i = \arccos(c_z/c) \Rightarrow \Omega, i \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} r \cos(V) \\ r \sin(V) \\ 0 \end{pmatrix} & = r \begin{pmatrix} \cos \Omega \cos(\omega + V) - \sin \Omega \sin(\omega + V) \cos i \\ \sin \Omega \cos(\omega + V) + \cos \Omega \sin(\omega + V) \cos i \\ \sin(\omega + V) \sin i \end{pmatrix} \Rightarrow \omega + V \\
r = \frac{p}{1 + e \cos(V)} & \Rightarrow \omega, V \\
\tan(E/2) = \frac{1 - e}{1 + e}^{1/2} \tan(V/2) & ; \quad M = E - e \sin E \Rightarrow M 
\end{align*}
\]

Calculation of position and velocity from osculating orbital elements (\textit{osc2rv.f})

\[
\begin{align*}
(a, e, i, \Omega, \omega, T; t) & \Rightarrow (x, y, z, v_x, v_y, v_z) \\
t & \Rightarrow M \quad E \quad (r, V) \\
M = n(t - T) & \quad M = E - e \sin E \\
r = a(1 - e \cos E) & \\
\tan(V/2) = \frac{1 + e}{1 - e}^{1/2} \tan(E/2) \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R \begin{pmatrix} r \cos(V) \\ r \sin(V) \\ 0 \end{pmatrix} : \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \frac{na^2}{r} \left\{ \vec{Q}(1 - e^2)^{1/2} \cos E - \vec{P} \sin E \right\} 
\end{align*}
\]

Where:

\[
R = \begin{pmatrix} R_3(-\Omega)R_1(-i)R_3(-\omega) = \\
R_3(-\Omega) & = \begin{pmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
R_1(-i) & = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix} \\
R_3(-\omega) & = \begin{pmatrix} P_x & Q_x & S_x \\ P_y & Q_y & S_y \\ P_z & Q_z & S_z \end{pmatrix} = [\vec{P} \vec{Q} \vec{S}] 
\end{align*}
\]
Exercise: Orbital elements variation:


a) Use program “rv2osc” to compute the instantaneous orbital elements $(X, Y, Z, Vx, Vy, Vz) \Rightarrow (a, e, i, \Omega, \omega, V)$

b) Plot the orbital elements in function of time to show their variation: \( a(t), e(t), i(t), \Omega(t), \omega(t), V(t) \)

c) Compare with the broadcast orbital elements

Solution:

a) cat 1995-10-18.eci|rv2osc> orb.dat

b) See the following plots
Contents

1. Elliptic orbit: Keplerian elements.
2. Perturbed Keplerian orbits: Osculating orbit.
3. GPS satellite coordinates computation and accuracy
   3.1. From Broadcast Navigation Message.
   3.2. From precise products.
4. GPS Satellite clock computation and accuracy
   4.1. From Broadcast Navigation Message.
   4.2. From precise products.
5. Geographic decorrelation of ephemeris errors.
Subframe 1 contains information about the parameters to be applied to satellite clock status for its correction. These values are polynomial coefficients that allow time onboard to be converted to GPS time. The subframe also contains information on satellite health condition.

Subframes 2 and 3 contain satellite ephemerides.

Subframe 4 provides ionospheric model parameters (in order to adjust for ionospheric refraction), UTC information, part of the almanac, and indications whether the A/S is activated or not (which transforms the P code into encrypted Y code).

Subframe 5 contains data from the almanac and on constellation status. It allows rapid identification of the satellite from which the signal comes. A total of 25 frames are needed to complete the almanac.
In order to calculate WGS84 satellite coordinates, you should apply the following algorithm [GPS/SPS-SS, table 2-15] (see in the book FORTRAN subroutine orbit.f)

### Ephemeris in navigation message

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_0$</td>
<td>Ephemerides reference epoch</td>
</tr>
<tr>
<td>$\sqrt{a}$</td>
<td>Square root of semi-major axis</td>
</tr>
<tr>
<td>$e$</td>
<td>Eccentricity</td>
</tr>
<tr>
<td>$M_0$</td>
<td>Mean anomaly at reference epoch</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Argument of perigee</td>
</tr>
<tr>
<td>$i_0$</td>
<td>Inclination at reference epoch</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>Ascending node’s right ascension</td>
</tr>
<tr>
<td>$\Delta n$</td>
<td>Mean motion difference</td>
</tr>
<tr>
<td>$\dot{i}$</td>
<td>Rate of inclination angle</td>
</tr>
<tr>
<td>$\dot{\Omega}$</td>
<td>Rate of node’s right ascension</td>
</tr>
<tr>
<td>$c_{MC}$, $c_{US}$</td>
<td>Latitude argument correction</td>
</tr>
<tr>
<td>$c_{RC}$, $c_{RS}$</td>
<td>Orbital radius correction</td>
</tr>
<tr>
<td>$c_{IC}$, $c_{IS}$</td>
<td>Inclination correction</td>
</tr>
</tbody>
</table>

### RINEX ephemeris file

```
2 NAVIGATION DATA GPS

00/08/17 09:31:37
00/08/17 09:31:37

gAGE BROADCAST EPHERIMES FILE
+1.769E+08 +2.352E+00 -1.1921E-07 -1.921E-07
+1.878E+05 +1.474E+05 -1.3017E+05 -3.2768E+05
+1.95777406693E+08 +1.59872115546E+14 405504

13 104 DELTA:UTC: A0, A1, T, W

LEAP SECONDS

END OF HEADER
```

---

Master of Science in GNSS

@ J. Sanz & J.M. Juan
3.1. Computation of satellite coordinates from navigation message (orbit.f)

- Computation of \( t_k \) time since ephemerids reference epoch \( t_{oe} \) (\( t \) and \( t_{oe} \) are given in GPS seconds of week):
  \[ t_k = t - t_{oe} \]

- Computation of mean anomaly \( M_k \) for \( t_k \):
  \[ M_k = M_0 + \left( \frac{\sqrt{\mu}}{a^3} + \Delta n \right) t_k \]

- Iterative resolution of Kepler’s equation in order to compute eccentric anomaly \( E_k \):
  \[ M_k = E_k - e \sin E_k \]

- Calculation of true anomaly \( v_k \):
  \[ v_k = \arctan \left( \frac{\sqrt{1 - e^2} \sin E_k}{\cos E_k - e} \right) \]

- Computation of latitude argument \( u_k \) from perigee argument \( W \), true anomaly \( v_k \) and corrections \( c_{uc} \) and \( c_{us} \):
  \[ u_k = \omega + v_k + c_{uc} \cos 2(\omega + v_k) + c_{us} \sin 2(\omega + v_k) \]

- Computation of radial distance \( r_k \), taking into consideration corrections \( c_{rc} \) and \( c_{rs} \):
  \[ r_k = a \left( 1 - 2 \cos E_k \right) + c_{rc} \cos 2(\omega + v_k) + c_{rs} \sin 2(\omega + v_k) \]

- Calculation of orbital plane inclination \( i_k \) from inclination \( i_o \) at reference epoch \( t_{oe} \) and corrections \( c_{ic} \) and \( c_{is} \):
  \[ i_k = i_o + i t_k + c_{ic} \cos 2(\omega + v_k) + c_{is} \sin 2(\omega + v_k) \]

- Computation of ascending node longitude \( \Omega_k \) (Greenwich), from longitude \( \Omega_o \) at start of GPS week, corrected from apparent variation of sidereal time at Greenwich between start of week and and reference time \( t_k = t - t_{oe} \) and also corrected from change of ascending node longitude since reference epoch \( t_{oe} \):
  \[ \Omega_k = \Omega_o + (\Omega - \Omega_E) t_k - \omega_E t_{oe} \]

- Calculation of coordinates in CTS system, applying three rotations (around \( u_k \), \( i_k \), \( \Omega_k \)):
  \[
  \begin{bmatrix}
  X_k \\
  Y_k \\
  Z_k
  \end{bmatrix} = R_3(-\Omega_k)R_1(-i_k)R_3(-u_k) \begin{bmatrix}
  r_k \\
  0 \\
  0
  \end{bmatrix}
  \]
Conventional Terrestrial Reference System (TRS):

**Earth Centered, Earth-Fixed (ECEF):**
the reference system rotates with Earth.

Reference values precise IGS orbits

Discrepancy between CODE and IGS combined product
Contents

1. Elliptic orbit: Keplerian elements.
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   4.2. From precise products.
5. Geographic decorrelation of ephemeris errors.
3.2 Computation of satellite coordinates from precise products.

Precise orbits for GPS satellites can be found on the International GNSS Service (IGS) server [http://igscb.jpl.nasa.gov](http://igscb.jpl.nasa.gov)

Orbits are given by \((x,y,z)\) coordinates with a sampling rate of 15 minutes. The satellite coordinates between epochs can be computed by polynomial interpolation. A 10th-order polynomial is enough for a centimetre level of accuracy with 15 min data.

\[
P_n(x) = \sum_{i=1}^{n} y_i \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)} = \frac{y_1}{x_1 - x_2} \cdots \frac{x - x_n}{x_1 - x_n} + \cdots + \frac{y_n}{x_n - x_1} \cdots \frac{x - x_i}{x_n - x_n-1}
\]

IGS orbit and clock products (for PPP):
Discrepancy between CODE and IGS combined product.
1. Elliptic orbit: Keplerian elements.
2. Perturbed Keplerian orbits: Osculating orbit.
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GPS Satellite Clock computation: Broadcast message

\[ dt_{sat} = a_0 + a_1(t-t_0) + a_2(t-t_0)^2 \]
Precise clocks for GPS satellites can be found on the International GNSS Service (IGS) server http://igscb.jpl.nasa.gov

They are providing precise orbits and clock files with a sampling rate of 15 min, as well as precise clock files with a sample rate of 5 min and 30 s in SP3 format.

Some centres also provide GPS satellite clocks with a 5 s sampling rate, like the les obtained from the Crustal Dynamics Data Information System (CDDIS) site.

Stable clocks with a sampling rate of 30 s or higher can be interpolated with a first-order polynomial to a few centimetres of accuracy. Clocks with a lower sampling rate should not be interpolated, because clocks evolve as random walk processes.
Precise Clock Interpolation: 300s samples
Precise Clock Interpolation: 30s samples

Session 3.2, Ex7b: Precise 30s clock interpolation error

Selective Availability (S/A): Intentional degradation of satellite clocks and broadcast ephemeris. (from 25 March, 1990)

GPS Before and After S/A was switched off

S/A was switched off at 2nd May 2000 and Permanently removed in 2008
### IGS Precise orbit and clock products:
RMS accuracy, latency and sampling

<table>
<thead>
<tr>
<th>Products (delay)</th>
<th>Broadcast (real time)</th>
<th>Ultra-rapid Predicted (real time)</th>
<th>Observed (3–9 h)</th>
<th>Rapid (17–41 h)</th>
<th>Final (12–18 d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orbit GPS (sampling)</td>
<td>~100 cm (~2 h)</td>
<td>~5 cm (15 min)</td>
<td>~3 cm (15 min)</td>
<td>~2.5 cm (15 min)</td>
<td>~2.5 cm (15 min)</td>
</tr>
<tr>
<td>Glonass (sampling)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>~5 cm (15 min)</td>
</tr>
<tr>
<td>Clock GPS (sampling)</td>
<td>~5 ns (daily)</td>
<td>~3 ns (15 min)</td>
<td>~150 ps (15 min)</td>
<td>~75 ps (5 min)</td>
<td>~75 ps (30 s)</td>
</tr>
<tr>
<td>Glonass (sampling)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>~TBD (15 min)</td>
</tr>
</tbody>
</table>

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Ephemeris Errors and Geographic decorrelation

- True position
- Reference Station
- Satellite location error
- Differential range error due to satellite orbit error
  \[ \delta \rho = \mathbf{e} \cdot \frac{\tilde{p}_{\text{user}}}{\rho_{\text{user}}} - \mathbf{e} \cdot \frac{\tilde{p}_{\text{ref}}}{\rho_{\text{ref}}} \]
- A conservative bound:
  \[ \delta \rho < \frac{b}{\rho} \varepsilon \]
  with a baseline \( b = 20km \)
  \[ \delta \rho < \frac{20}{20000} \varepsilon = \frac{1}{1000} \varepsilon \]
Range error from CREU and EBRE

Differential range error from between CREU and EBRE

\[ \delta \rho = \tilde{e} \cdot \tilde{\rho}_{\text{user}} - \tilde{e} \cdot \tilde{\rho}_{\text{ref}} \]

288 km of baseline

CREU-EBRE: 288 km; SPP: 2000m Along-Track orbit error

Differential positioning

CREU Absolute positioning

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Errors over the hyperboloid (i.e. $\rho_b - \rho_a = ctt$) will not produce differential range errors.

The highest error is given by the vector $\hat{u}$, orthogonal to the hyperboloid and over the plain containing the baseline vector $\hat{b}$ and the LoS vector $\hat{p}$.

Note: Being the baseline $\hat{b}$ much smaller than the distance to the satellite, we can assume that the LoS vectors from A and B receive are essentially identical to $U$.

That is, $\rho_b \approx \rho_a = \rho$.

$$
\begin{align*}
\hat{u} &= \hat{p} \times (\hat{b} \times \hat{p}) = \hat{b} (\hat{p}^T \cdot \hat{p}) - \hat{p} (\hat{p}^T \cdot \hat{b}) \\
&= I \hat{b} - (\hat{\rho} \cdot \hat{p}^T) \hat{b} = (I - \hat{\rho} \cdot \hat{p}^T) \hat{b}
\end{align*}
$$

Note: $\hat{u} = \sin \phi \hat{\rho}$

Note: being $\hat{u}$ a vector orthogonal to the LoS $\hat{p}$, thence, $\hat{e}_1 = \hat{r} \hat{u}$

Thence:

$$
\begin{align*}
\delta \rho &= - \frac{b \sin \phi}{\rho} \hat{e} \cdot \hat{u} = - \hat{e} \cdot (\sin \phi \hat{\rho}) \frac{b}{\rho} \\
&= - \hat{e} \left( (1 - \hat{\rho} \cdot \hat{p}^T) \frac{b}{\rho} \right) \\
\end{align*}
$$

Where: $\hat{b} = b \hat{\hat{b}}$

is the baseline vector.

---

**ORBIT TEST:**

Broadcast orbits

Along-track Error (PRN17)

**PRN17:**

Doy=077, Transm. time: 64818 sec

---

<table>
<thead>
<tr>
<th>17</th>
<th>10</th>
<th>3</th>
<th>18</th>
<th>20</th>
<th>0</th>
<th>0.0</th>
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<th>2.842170943040E-12</th>
<th>0.000000000000E+00</th>
<th>7.800000000000E+01</th>
<th>5.059375000000E-01</th>
<th>4.000000000000E+00</th>
</tr>
</thead>
<tbody>
<tr>
<td>-9.257976353169E-05</td>
<td>5.277505260892E-03</td>
<td>8.186325430870E-06</td>
<td>5.153578153610E+03</td>
<td>4.176000000000E+05</td>
<td>-9.800000000000E+01</td>
<td>5.893750000000E-01</td>
<td>4.506974472820E-09</td>
<td>2.842170943040E-12</td>
<td>0.000000000000E+00</td>
<td>7.800000000000E+01</td>
<td>5.059375000000E-01</td>
<td>4.000000000000E+00</td>
</tr>
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<td>2.547856603660E+00</td>
<td>7.964974683078E-09</td>
<td>2.215312500000E+02</td>
<td>2.547856603660E+00</td>
<td>7.964974683078E-09</td>
<td>2.215312500000E+02</td>
<td>2.547856603660E+00</td>
</tr>
<tr>
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<td>4.000000000000E+05</td>
<td>4.000000000000E+05</td>
<td>4.000000000000E+05</td>
</tr>
</tbody>
</table>

---

**diff EPH.dat.org EPHcuc_x0.dat**

---

By Miguel Juan Zornoza

\[ a = \frac{(\rho_b - \rho_a)}{2} : \text{ hyperboloid semiaxis} \]

\[ b = \frac{1}{2} : \text{ focal length} \]

where \( a = b / 2 \cdot \cos(\phi) \)

Note: in this 3D problem \( \phi \) is NOT the elevation of ray.
Exercise:

Justify that clock errors completely cancel in differential positioning.
ERRORS on the Signal

• Space Segment Errors:
  - Clock errors
  - Ephemeris errors

  Common

• Propagation Errors
  - Ionospheric delay
  - Tropospheric delay

  Strong spatial correlation

• Local Errors
  - Multipath
  - Receiver noise

  Weak spatial correlation

  No spatial correlation

References


Thank you
Lecture 3
Position estimation with pseudoranges

Contact: jaume.sanz@upc.edu
Web site: http://www.gage.upc.edu

Authorship statement

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5 March 2017
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   1.2. Code measurements modelling
   1.3. Example of computation of modelled pseudorange

2. Linear observation model and parameter estimation
   2.1. Least Squares solution (conceptual view)
   2.2. Weighted Least Squares and Minimum Variance estimator
   Example of solution computation
   2.3. Kalman Filter (conceptual view)
   Examples of static and kinematic positioning

---

Introduction: Linear model and Prefit-residuals

**Input:**
- **Pseudoranges** (receiver-satellite j): \( P_j \)
- Navigation message. In particular:
  - **Satellite position** when transmitting signal: \( r^j = (x^j, y^j, z^j) \)
  - **Offsets** of satellite clocks: \( d^j t \)
    \( (j = 1,2,...,n) \quad (n \geq 4) \)

**Unknowns:**
- **Receiver position**: \( r = (x, y, z) \)
- **Receiver clock offset**: \( DT \)
GNSS positioning concept

- GNSS uses technique of "triangulation" to find user location
- To "triangulate" a GNSS receiver needs:
  - **To know the satellite coordinates** and clock synchronism errors:
    - Satellites broadcast orbits parameters and clock offsets.
  - **To measure distances from satellites**:
    - This is done measuring the traveling time of radio signals:
      ("Pseudo-ranges": Code and Carrier measurements)
    - Measurements must be corrected by several error sources:
      Atmospheric propagation, relativity, clock offsets, instrumental delays...

For each satellite in view

\[
C_{rec}^{sat} = \rho_{rec}^{sat} + c \cdot (dt_{rec} - dt_{sat}) + \sum \delta_k + \varepsilon
\]

Linearising \( \rho \) around an ‘a priori’ receiver position \((x_{0,rec}, y_{0,rec}, z_{0,rec})\)

\[
= \rho_{0,rec}^{sat} + \frac{x_{0,rec} - x_{sat}}{\rho_{0,rec}^{sat}} \Delta x_{rec} + \frac{y_{0,rec} - y_{sat}}{\rho_{0,rec}^{sat}} \Delta y_{rec} + \frac{z_{0,rec} - z_{sat}}{\rho_{0,rec}^{sat}} \Delta z_{rec} + c (dt_{rec} - dt_{sat}) + \sum \delta_k
\]

where:

\[
\Delta x_{rec} = x_{rec} - x_{0,rec} \quad ; \quad \Delta y_{rec} = y_{rec} - y_{0,rec} \quad ; \quad \Delta z_{rec} = z_{rec} - z_{0,rec}
\]

Prefit-residuals (Prefit)

\[
C_{rec}^{sat} - \rho_{0,rec}^{sat} - c dt_{sat} - \sum \delta_k = \frac{x_{0,rec} - x_{sat}}{\rho_{0,rec}^{sat}} \Delta x_{rec} + \frac{y_{0,rec} - y_{sat}}{\rho_{0,rec}^{sat}} \Delta y_{rec} + \frac{z_{0,rec} - z_{sat}}{\rho_{0,rec}^{sat}} \Delta z_{rec} + c dt_{rec}
\]

measurement | computed | unknown
For all satellites in view:

\[
\begin{bmatrix}
\text{Prefit}_1 \\
\text{Prefit}_2 \\
\text{......} \\
\text{Prefit}_n \\
\end{bmatrix}
\]

Observations (measured-computed)

Ununknowns

Measurements modelling:

Prefit residual is the difference between measured and modeled pseudorange:

\[
\text{Prefit}_\text{rec} = C_{\text{rec}1}^{\text{sat}}[\text{measured}] - C_{\text{rec}1}^{\text{sat}}[\text{modelled}]
\]

where:

\[
C_{\text{rec}1}^{\text{sat}}[\text{modelled}] = \rho_{\text{rec},0}^{\text{sat}} - c \left( \frac{d^{\text{sat}} + \Delta r^{\text{sat}}}{dt^{\text{sat}}} \right) + Trop_{\text{rec}}^{\text{sat}} + Ion_{\text{rec}}^{\text{sat}} + TGD^{\text{sat}}
\]

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Code Pseudorange modeling

The pseudorange modeling is based in the GPS Standard Positioning Service Signal Specification (GPS/SPS-SS).

\[
C_{1\text{sat}}^{\text{modelled}} = \rho_{\text{rec},0}^{\text{sat}} - c \left( \tilde{d}^{\text{sat}} + \Delta r^{\text{sat}} \right) + T_{\text{rec}}^{\text{sat}} + I_{\text{rec}}^{\text{sat}} + T_{\text{GD}}^{\text{sat}}
\]

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Euclidean distance between satellite coordinates at emission time and receiver coordinates at reception time.

\[ \rho_{0,\text{rec}} = \sqrt{(x_{\text{sat}} - x_{0,\text{rec}})^2 + (y_{\text{sat}} - y_{0,\text{rec}})^2 + (z_{\text{sat}} - z_{0,\text{rec}})^2} \]

Of course, receiver coordinates are not known (is the target of this problem). But ....

\[ C_{1,\text{rec}}^{\text{sat}}[^{\text{modelled}}] = \rho_{0,\text{rec}}^{\text{sat}} - c (d_{\text{sat}}^{\text{sat}} + \Delta \text{rel}^{\text{sat}} + \text{Trop}^{\text{sat}}_{\text{rec}} + \text{Ion}^{\text{sat}}_{\text{rec}} + \text{TGD}^{\text{sat}}) \]

Thence, the navigation problem will consist on:

1.- To start from an approximate value for receiver position \((x_{0,\text{rec}}, y_{0,\text{rec}}, z_{0,\text{rec}})\) e.g. the Earth’s centre to linearise the equations.
2.- With the pseudorange measurements and the navigation equations, compute the correction \((\Delta x_{\text{rec}}, \Delta y_{\text{rec}}, \Delta z_{\text{rec}})\) to have improved estimates:

\[ (x_{\text{rec}}, y_{\text{rec}}, z_{\text{rec}}) = (x_{0,\text{rec}}, y_{0,\text{rec}}, z_{0,\text{rec}}) + (\Delta x_{\text{rec}}, \Delta y_{\text{rec}}, \Delta z_{\text{rec}}) \]

3.- Linearise the equations again, about the new improved estimates, and iterate until the change in the solution estimates is sufficiently small.

The estimates converges quickly. Generally in two to four iterations, even if starting from the Earth’s Centre.
Satellite coordinates at **emission time** $(\text{rec2ems.f})$

- The GPS signal travels from satellite coordinates at emission time $(T_{\text{emis}})$ to receiver coordinates at reception time $(T_{\text{recep}})$.
- The satellite can move several hundreds of meters from $T_{\text{emis}}$ to $T_{\text{recep}}$.

The receiver time-tags are given at reception time and in the receiver clock time.

An algorithm is needed to compute the satellite coordinates at **emission time** “in the GPS system time” from reception time in the receiver time tags.

As it is known, the pseudorange measurements link the “emission time $(T_{\text{emis}})$” in satellite clock $(t^S)$ with reception time $(T_{\text{recep}})$ in receiver clock $(t_R)$ (receiver time tags).

Thence, the emission time in the satellite clock is:

$$t^S(T_{\text{emis}}) = t_R(T_{\text{recep}}) - C_1/c$$

Finally, since $dt^S = t^S - T$ is the time offset between satellite clock $(t^S)$ and **GPS system time** $(T)$, thence:

$$T_{\text{emis}} = t^S(T_{\text{emis}}) - dt^S = t_R(T_{\text{recep}}) - C_1/c - dt^S$$
Distance: $\Delta r$

Variation in range: $\Delta \rho = \rho_{\text{emission}} - \rho_{\text{reception}}$

Note: $\rho_{\text{reception}}$ is computed unsetting in gLAB:
- Satellite movement during signal flight time.
- Earth rotation during signal flight time.
Vertical error comparison

Horizontal error comparison
The algorithm provided by the GPS/SPS-SS (\texttt{orbit.f}) supplies satellite coordinates in an Earth-Fixed reference frame. To compute the coordinates at the emission time, the following algorithm can be applied:

1. From receiver time-tags, compute emission time in GPS system time:
   \[ T_{\text{emis}} = t_R(T_{\text{recep}}) - \left( \frac{C_1}{c} + d t^S \right) \]

2. Compute satellite coordinates at emission time \( T_{\text{emis}} \)
   \[
   T_{\text{emis}} \rightarrow [\text{orbit}] \rightarrow (X_{\text{sat}}, Y_{\text{sat}}, Z_{\text{sat}})_{\text{CTS}[\text{emission}]}
   \]

3. Account for Earth rotation during traveling time from emission to reception \( \Delta t \) (CTS reference system at reception time is used to build the navigation equations).
   \[
   (X_{\text{sat}}, Y_{\text{sat}}, Z_{\text{sat}})_{\text{CTS[reception]}} = R_3(\omega_E \Delta t). (X_{\text{sat}}, Y_{\text{sat}}, Z_{\text{sat}})_{\text{CTS[emission]}}
   \]

Variation in range: \( \Delta \rho = \rho' - \rho_{\text{emission}} \)
Vertical error comparison

Horizontal error comparison
Satellite and receiver clock offsets

- They are time-offsets between satellite/receiver time and GPS system time (provided by the ground control segment):

- The receiver clock offset \( dt_{rec} \) is estimated together with receiver coordinates.

- Satellite clock offset \( dt_{sat} \) may be computed from navigation message plus a Relativistic clock correction

\[
dt_{sat} = a_0 + a_1(t-t_0) + a_2(t-t_0)^2 + \Delta rel_{sat}
\]

\[
C1^{sat}_{rec}[modelled] = \rho_{rec,0}^{sat} - c dt_{sat}^{sat} + \Delta rel_{sat}^{sat} + \text{Trop}_{rec}^{sat} + \text{Ion}_{rec}^{sat} + \text{TGD}_{sat}
\]
Range variation: satellite clocks

Vertical error comparison
Relativistic clock correction ($\Delta_{rel}$)

- A constant component depending only on nominal value of satellite’s orbit major semi-axis, being corrected modifying satellite’s clock oscillator frequency*:

$$\frac{f' - f_0}{f_0} = \frac{1}{2} \left( \frac{v}{c} \right)^2 + \Delta\frac{U}{c^2} = -4.464 \cdot 10^{-10}$$

- A periodic component due to orbit eccentricity (to be corrected by user receiver):

$$\Delta_{rel} = -2 \sqrt{\frac{\mu a}{c^2}} e \sin(E) = -2 \frac{r \cdot v}{c^2} \text{ (seconds)}$$

Being $\mu = 3.986005 \cdot 10^{14} \text{ (m}^3/\text{s}^2)$ universal gravity constant, $c = 299792458 \text{ (m/s)}$ light speed in vacuum, $a$ is orbit’s major semi-axis, $e$ is its eccentricity, $E$ is satellite’s eccentric anomaly, and $r$ and $v$ are satellite’s geocentric position and speed in an inertial system.

*being $f_0 = 10.23 \text{ MHz}$, we have $\Delta f = 4.464 \cdot 10^{-10} f_0 = 4.57 \cdot 10^{-3} \text{ Hz}$ so satellite should use $f'o = 10.2299999543 \text{ MHz}$. 
Range variation: relativistic correction

Barcelona, Spain: 2005 5 29

Vertical error comparison

Barcelona, Spain: 2005 5 29
Receiver: NovAtel OEM3, Antenna: NovAtel 800 (Pinwheel)

Relativistic correction = NO
Relativistic correction = YES
**Ionospheric Delay**  \( I_{\text{on}}^{\text{sat}}_{\text{rec}} \)

The ionosphere extends from about 60 km in height until more than 2000 km, with a sharp electron density maximum at around 350 km. The ionosphere delays code and advances carrier by the same amount.

The ionospheric delay depends on signal frequency as given by:

\[
I_{\text{on}}^{\text{sat}}_{\text{rec}} = \frac{40.3}{f_i^2} I
\]

Where \( I \) is number of electrons per area unit in the direction of observation, or STEC (Slant Total Electron Content)

\[
I = \int_{\text{rec}}^{\text{sat}} N_e ds
\]

- For two-frequency receivers, it may be cancelled (99.9%) using ionosphere-free combination

\[
LC = \frac{f_1^2 L_1 - f_2^2 L_2}{f_1^2 - f_2^2}
\]

- For one-frequency receivers, it may be corrected (about 60%) using Klobuchar model (defined in GPS/SPS-SS), whose parameters are sent in navigation message. (See program klob.f)
The Klobuchar model (klob.f) was designed to minimize user computational complexity.

- Minimum user computer storage
- Minimum number of coefficients transmitted on satellite-user link
- At least 50% overall RMS ionospheric error reduction worldwide.

It is assumed that the electron content is concentrated in a thin layer at 350 km in height.

- The slant delay is computed from the vertical delay at the Ionospheric Pierce Point (IPP), multiplying by the obliquity factor.

\[
\text{Ion}_{\text{slant}} = \text{Ion}_{\text{vert}} m(\text{elev})
\]

\[
m(\text{elev}) = \left[ 1 - \left( \frac{R_E}{R_E + h} \cos(\text{elev}) \right)^2 \right]^{-1/2}
\]
Being:

\[ A = \sum_{n=0}^{3} \alpha_n \phi^n \quad ; \quad P = \sum_{n=0}^{3} \beta_n \phi^n \]

\[ \phi = \text{Geomagnetic Latitude} \]

Where:

\[ Dc = 5\text{ns} \quad \text{(c.t.t. phase offset)} \]

\[ t = \text{Local Time} \]

\[ \text{ Ion}_{\text{SLANT}} = \text{ Ion}_{\text{VERT}} \cdot m(\text{elev}) \]

\[ m(\text{elev}) = \left[ 1 - \left( \frac{R_E}{R_E + h} \cos(\text{elev}) \right) \right]^{1/2} \]

Local Time (hours)

![Amplitude vs Local Time](image)

**Klobuchar model**

\[ \text{Ion}_{\text{VERT}} = \begin{cases} 
DC + A \cos \left( \frac{2\pi(t - \Phi)}{P} \right) & \text{(day)} \\
DC \quad \text{if} \quad \left( \frac{2\pi(t - \Phi)}{P} \right) > \frac{\pi}{2} & \text{(night)}
\end{cases} \]

**RINEX data example**

```
2 NAVIGATION DATA
CCRINEXN V1.5.2 UX CDDIS IGS BROADCAST EPHEMERIS FILE

24-MAR-2000 00:23 PGM / RUN BY / DATE

0.3167D-07 0.4051D-07 -0.2347D-06 0.1732D-06 ION ALPHA
-0.2842D+05 -0.2150D+05 -0.1096D+06 0.4301D+06 ION BETA
-0.121071934700D-07 -0.488498130835D-13 319488 \[\text{DELTA-UTC: A0,A1,T,W}\]
0.191000000000D+03 -0.106250000000D+01 0.487163149444D-08 -0.123716752769D+01
-0.540167093277D-07 0.476544268895D-02 0.713579356670D-05 0.515433833885D+04
0.172800000000D+06 -0.260770320892D-07 -0.850753478531D+00 0.763684511185D-07
0.957259887797D+00 0.241437500000D+03 -0.167990552187D+01 -0.823998688564D-08
0.174650132022D-09 0.100000000000D+01 0.100200000000D+04 0.000000000000D+00
0.320000000000D+02 0.000000000000D+00 0.465661287308D-09 0.191000000000D+03
0.172800000000D+06 0.000000000000D+00 0.000000000000D+00 0.000000000000D+00

END OF HEADER
```

Master of Science in GNSS
Range variation: Ionospheric correction

Klobuchar model

\[ \text{Ion}_{\text{SLANT}} = \text{Ion}_{\text{VERT}} \cdot m(\text{elev}) \]

\[ m(\text{elev}) = \left[ 1 - \left( \frac{R_E}{R_E + h \cos(\text{elev})} \right)^2 \right]^{-1/2} \]

Vertical error comparison

Barcelona, Spain: 2005 5 29
Receiver: NovAtel OEM3, Antenna: NovAtel 600 (Pinnacle)

Ionospheric correction = NO
Ionospheric correction = YES
Galileo Single Frequ. Ionospheric Corr. Algo. (NeQuick model)

1. Observe slant TEC in Galileo Sensor Stations for 24 hours
2. Transmit effective ionisation parameter in Galileo Navigation message
   \[ Az = a_0 + a_1 \cdot \mu + a_2 \cdot \mu^2 \]
3. Calculate slant TEC using NeQuick with broadcast ionisation parameter.
   Correct for Ionospheric delay at frequency in question.

\( \mu \) is the Modified DIP latitude (MODIP)

Source: Bertram Arbesser-Rastburg
ESA/JRC International Summer School on GNSS 2016
The MODI P bounds are represented graphically, showing the variation of the dip (dip latitude) across different longitudes (LHT).

**MODI P bounds**

The equation given is:

\[
\tan \mu = \frac{I}{\sqrt{\cos \varphi}}
\]

where \( I \) is the true magnetic inclination, or dip in the ionosphere (usually at 300 km), and \( \varphi \) is the geographic latitude of the receiver.

**Ionospheric models used by the GNSSs**

<table>
<thead>
<tr>
<th>GNSS</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>GPS</td>
<td>Klobuchar model</td>
</tr>
<tr>
<td>GLONASS</td>
<td>No ionospheric model is broadcasted</td>
</tr>
<tr>
<td>BeiDou</td>
<td>Klobuchar model (with layer height at 375km instead of 350km)</td>
</tr>
<tr>
<td>Galileo</td>
<td>NeQuick model</td>
</tr>
</tbody>
</table>
Ionospheric models performance comparison

![Graph showing ionospheric models performance comparison](Picture from [RD-5])

- Klobuchar GPS Model
- NeQuick GAL Model
- IGS Rapid GIM (2h)
- Klobuchar GPS Model
- NeQuick GAL Model
- IGS Rapid GIM (2h)
- Fast-PPP GIM (15 min)
- Fast-PPP (5 min)

Time (Month Of Year 2014)

Daily RMS Post-fit (TECUs)
Tropospheric Delay

Troposphere is the atmospheric layer placed between Earth’s surface and an altitude of about 60km.

The tropospheric delay does not depend on frequency and affects both the code and carrier phases in the same way. It can be modeled (about 90%) as:

- $d_{\text{dry}}$ corresponds to the vertical delay of the dry atmosphere (basically oxygen and nitrogen in hydrostatic equilibrium)
  - It can be modeled as an ideal gas.
- $d_{\text{wet}}$ corresponds to the vertical delay of the wet component (water vapor) → difficult to model.

A simple model is:

$$Trop_{\text{rec}}^{\text{sat}} = (d_{\text{dry}} + d_{\text{wet}}) \cdot m(\text{elev})$$

$$d_{\text{dry}} = 2.3\exp(-0.116 \cdot 10^{-3} H) \text{ meters}$$

$$d_{\text{wet}} = 0.1m \quad [H: \text{height over the sea level}]$$

$$m(\text{elev}) = \frac{1.001}{\sqrt{0.002001 + \sin^2(\text{elev})}}$$

$$C1_{\text{rec}}^{\text{sat}} [\text{modelled}] = \rho_{0,\text{rec}}^{\text{sat}} - c \left( d\tilde{I}_{\text{sat}} + \Delta \text{rel}^{\text{sat}} \right) + Trop_{\text{rec}}^{\text{sat}} + Ion_{1,\text{rec}}^{\text{sat}} + \text{TGD}^{\text{sat}}$$

---

![Troposphere: slant factor](image)
Range variation: Tropospheric correction

Barcelona, Spain: 2005 5 29

Tropospheric correction

RTCA-Do229C model

Vertical error comparison

Barcelona, Spain: 2005 5 29
Receiver: NovAtel OEMS, Antenna: NovAtel 800 (Pinwheel)

Tropospheric correction = NO
Tropospheric correction = YES
**Instrumental Delays**

Some sources for these delays are antennas, cables, as well as several filters used in both satellites and receivers.

They are composed by a delay corresponding to satellite and other to receiver, depending on frequency:

- \( K_{1,\text{rec}}^{\text{sat}} = K_{1,\text{rec}} + TGD_{\text{sat}} \)
- \( K_{2,\text{rec}}^{\text{sat}} = K_{2,\text{rec}} + \frac{f_1}{f_2} TGD_{\text{sat}} \)

- \( K_{1,\text{rec}} \) may be assumed as zero (including it in receiver clock offset).
- \( TGD_{\text{sat}} \) is transmitted in satellite’s navigation message (Total Group Delay).

According to ICD GPS-2000, control segment monitors satellite timing, so TGD cancels out when using free-ionosphere combination. That is why we have that particular equation for \( K_{2} \).

\[
C_{\text{rec}}^{\text{sat}} \text{[modelled]} = \rho_{0,\text{rec}}^{\text{sat}} - c \left( \Delta t_{\text{sat}} + \Delta rel_{\text{sat}} \right) + Trop_{\text{rec}}^{\text{sat}} + Ion_{\text{rec}}^{\text{sat}} + TGD_{\text{sat}}
\]
Range variation: Instrumental delays (TGD)

Vertical error comparison
Horizontal error comparison

Barcelona, Spain: 2005 5 29  
Receiver: NovAtel OEM3, Antenna: NovAtel 800 (Pinwheel)

Measurement noise (thermal noise)

**Antispoofing (A/S):**
The code P is encrypted to Y. ➔ Only the code C at frequency L1 is available.

<table>
<thead>
<tr>
<th>Code measurements</th>
<th>Wavelength (chip-length)</th>
<th>σ noise (1% of λ) [*]</th>
<th>Main characteristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>300 m</td>
<td>3 m</td>
<td>Unambiguous but noisier</td>
</tr>
<tr>
<td>P1 (Y1): encrypted</td>
<td>30 m</td>
<td>30 cm</td>
<td></td>
</tr>
<tr>
<td>P2 (Y2): encrypted</td>
<td>30 m</td>
<td>30 cm</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Phase measurements</th>
<th>L1</th>
<th>19.05 cm</th>
<th>2 mm</th>
<th>Precise but ambiguous</th>
</tr>
</thead>
<tbody>
<tr>
<td>L2</td>
<td>24.45 cm</td>
<td>2 mm</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

[*] codes may be smoothed with the phases in order to reduce noise (i.e., C1 smoothed with L1 ➔ 50 cm noise)
Multipath

- One or more reflected signals reach the antenna in addition to the direct signal. Reflective objects can be earth surface (ground and water), buildings, trees, hills, etc.
- It affects both code and carrier phase measurements, and it is more important at low elevation angles.

![Diagram of Multipath](image)

- **Code:** up to 1.5 chip-length ➔ up to 450m for C1 [theoretically]
  Typically: less than 2-3 m.
- **Phase:** up to $\lambda/4$ ➔ up to 5 cm for L1 and L2 [theoretically]
  Typically: less than 1 cm

---

Receiver and multipath noise

![Graph of Receiver and multipath noise](image)

GPS standalone (C1 code)

**10,000 €**
Receiver noise and multipath

GPS standalone (C1 code)  
100 €

Contents

1. Measurements modelling and error sources
   1.1. Introduction: Linear model and Prefit-residual
   1.2. Code measurements modelling
   1.3. Example of computation of modelled pseudorange

2. Linear observation model and parameter estimation
   2.1. Least Squares solution (conceptual view)
   2.2. Weighted Least Squares and Minimum Variance estimator
       Example of solution computation
   2.3. Kalman Filter (conceptual view)
       Examples of static and kinematic positioning
Example of Computation of modeled pseudorange

Using data of files **gage2860.98o** and **brdc2860.98n**, compute “by hand” the modeled pseudorange for satellite PRN 14 at t=38230 sec (10h37m10s).

\[ C_{\text{rec [modelled]}}^{\text{sat}} = \rho_{0, \text{sat}}^{\text{rec}} - c \left( d_{\text{sat}}^{\text{rec}} + \Delta r_{\text{rel}}^{\text{sat}} \right) + Trop_{\text{rec}}^{\text{sat}} + Ion_{\text{rec}}^{\text{sat}} + TGD_{\text{sat}} \]

Follow these steps:

1. Select orbital elements closer to 38230
2. Compute satellite clock offset
3. Compute satellite-receiver aprox. geometric range
   3.1 Compute emission time from receiver (reception) time-tags and code pseudorange.
   3.2 Compute satellite coordinates at emission time
   3.3 Compute approximate geometric range.
4. Compute satellite Instrumental delay (TGD):
5. Compute relativistic satellite clock correction
6. Compute tropospheric delay
7. Compute ionospheric delay
8. Compute modeled pseudorange from previous values:

\[ C_{\text{rec [modelled]}}^{\text{sat}} = \rho_{0, \text{sat}}^{\text{rec}} - c \left( d_{\text{sat}}^{\text{rec}} + \Delta r_{\text{rel}}^{\text{sat}} \right) + Trop_{\text{rec}}^{\text{sat}} + Ion_{\text{rec}}^{\text{sat}} + TGD_{\text{sat}} \]

See also exercise 5, Session 5.2 in [RD-2]
1. Selection of orbital elements: From file \textit{brdc2860.98n}, select the last transmitted navigation message block before instant t=38230 s (10h37m10s).

Transmission time: 
979 208818 \rightarrow 10\text{h} 0\text{m} 18\text{s}

2. Satellite clock offset computation: From file \textit{brdc2860.98n}, compute satellite clock offset at time t=3830 s for PRN14:

\[ t = 38230 \text{ sec} \]
\[ t_0 = 12\text{h} 0\text{m} 0\text{s} = 43200 \text{ s} \]

\[ \Delta t_{\text{sat}} = a_0 + a_1 (t - t_0) + a_2 (t - t_0)^2 = 5.65 \cdot 10^{-6} \text{ s} \]

\[ C1_{\text{rec}[\text{modelled}]} = \rho_{0,\text{rec}} - c (\Delta t_{\text{sat}} + \Delta r_{\text{sat}}) + Trop_{\text{rec}}^{\text{sat}} + Ion_{\text{rec}}^{\text{sat}} + TGD_{\text{sat}} \]
3. Satellite-receiver geometric range computation:

Use the following values (4789031, 176612, 4195008) as approximate coordinates.

3.1: Emission time computation from receiver time-tag and code pseudorange:

\[ T_{\text{emis}} = t_R(T_{\text{recep}}) - \left( \frac{C1}{c} + dt^{\text{sat}} \right) \]

Measurement file gage2860.98o

Pseudorange \( C1 \) at receiver time-tag

t=38230sec: \( C1 = 23585247.703 \) m

Ephemeris file brdc2860.98n

Satellite clock offset at t=38230 sec

dt^{\text{sat}} = 5.65 \times 10^{-6} \) sec (see previous results)

Thence, the emission time in GPS satellite clock is:

\[ T_{\text{emis}} = 38230 - \left( \frac{23585247.703}{c} + 5.65 \times 10^{-6} \right) = 38229.921 \text{ sec} \]

(where \( c = 299792458 \) m/s)

Note:

From RINEX measurement file gage2860.98o, select the \( C1 \) pseudorange measurement at receiver time-tag for PRN14:

\[ \text{PRN 14} \]

\[ t = 38230 \text{ sec} = 10h\ 37m\ 10s \]

<table>
<thead>
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<th>4</th>
<th>10</th>
<th>13</th>
<th>10</th>
<th>37</th>
<th>10.0000000</th>
<th>0</th>
<th>5G18G14G16G</th>
<th>4G19</th>
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<tr>
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<td>0.000</td>
<td>20143892.105</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-220595.001</td>
<td>0.000</td>
<td>23585247.703</td>
<td>0.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1305085.999</td>
<td>0.000</td>
<td>23146887.826</td>
<td>0.000</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>6246118.999</td>
<td>0.000</td>
<td>20798091.711</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-19853878.999</td>
<td>0.000</td>
<td>22235319.057</td>
<td>0.000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Thence:

Measurement file gage2860.98o

Pseudorange \( C1 \) at receiver time-tag

t=38230sec: \( C1 = 23585247.703 \) m
3.2: Satellite coordinates at emission time pseudorange:

The previous coordinates are given in an Earth-Centered-Earth-Fixed reference frame (CTS) at $T_{\text{emission}} = 38229.921$ sec. This reference frame rotates by an amount $\omega_E \Delta t$ during traveling time $\Delta t = T_{\text{reception}} - T_{\text{emission}}$.

$$(X_{\text{sat}}, Y_{\text{sat}}, Z_{\text{sat}})_{\text{CTS[reception]}} = R_3(\omega_E \Delta t). (X_{\text{sat}}, Y_{\text{sat}}, Z_{\text{sat}})_{\text{CTS[emission]}}$$

$$\begin{bmatrix}
11453350.377 \\
122468589.797 \\
8245076.145
\end{bmatrix}_{\text{CTS[reception]}} = 
\begin{bmatrix}
\cos(\omega_E \Delta t) & \sin(\omega_E \Delta t) & 0 \\
-\sin(\omega_E \Delta t) & \cos(\omega_E \Delta t) & 0 \\
0 & 0 & 1
\end{bmatrix} 
\begin{bmatrix}
11453479.346 \\
122468524.004 \\
8245076.145
\end{bmatrix}_{\text{CTS[emission]}}$$

$\omega_E \Delta t = -5.74 \cdot 10^{-6} \text{ rad}$. (where $\omega_E = 7.2921151467 \cdot 10^{-5} \text{ rad/sec}$)

$$\Delta t = -\frac{\rho_{0, \text{rec}}}{c} = -0.079 \text{ sec}$$

$$\rho_{0, \text{rec}} = \sqrt{(x_{\text{sat}} - x_{0, \text{rec}})^2 + (y_{\text{sat}} - y_{0, \text{rec}})^2 + (z_{\text{sat}} - z_{0, \text{rec}})^2} \approx 23616673.3 \text{ m}$$

$(x, y, z)_{\text{satellite}} \approx (11453479, 122468524, 8245076)$

$(x_0, y_0, z_0)_{\text{receiver}} \approx (4789031, 176612, 4195008)$

An approximate value is enough to compute $\Delta t$.

Note: Both satellite and receiver coordinates must be given in the same reference system! ➔ the CTS[reception] will be used to build navigation equations.
3.2: Geometric range computation

The geometric range between satellite coordinates at emission time and the “approximate position of the receiver” at reception time (both coordinates given in the same reference system [for instance the CTS system at reception time]) is computed by:

\[
\rho_{0,receiver}^\text{sat} = \sqrt{(x_{0,receiver}^\text{sat} - x_{0,rec}^\text{sat})^2 + (y_{0,receiver}^\text{sat} - y_{0,rec}^\text{sat})^2 + (z_{0,receiver}^\text{sat} - z_{0,rec}^\text{sat})^2} = 23616699.124 m
\]

\[
(x, y, z)^\text{satellite} = (11453350.2771, 22468589.7975, 8245076.1448)^\text{CTS[reception]}
\]

\[
(x_0, y_0, z_0)^\text{receiver} = (4789031, 176612, 4195008)^\text{CTS[reception]}
\]

“Approximate” receiver coordinates at reception time.

\[
C^\text{sat}_{rec}[\text{modelled}] = \rho_{0,rec}^\text{sat} - c \left( \Delta t^\text{sat} + \Delta r^\text{sat}_1 \right) + Trop_{rec}^\text{sat} + Ion_{1,rec}^\text{sat} + TGD^\text{sat}
\]

4. Time Group Delay (TGD) or Satellite Instrumental delay.

⇒ From file brdc2860.98n, compute the TGD for PRN14:

\[
\begin{align*}
\text{TGD} &= -2.32830643654E-09 \times c = -0.69801 m
\end{align*}
\]
5. Relativistic clock correction:

\[
\Delta \text{rel}^{sat} = -2 \sqrt{\mu a \over c^2} \cdot e \sin(E) = -2.3 \cdot 10^{-10} \text{s}
\]

\[
C1_{rec}^{sat} [\text{modelled}] = \rho_{0,rec}^{sat} - c \left( d^{sat} + \Delta \text{rel}^{sat} \right) + Trop_{rec}^{sat} + Ion_{rec}^{sat} + TGD^{sat}
\]

Temission = 38229.921 s

\[ E = 0.095 \text{ rad (eccentric anomaly) } \]

\[ \mu = 3.986005 \times 10^{14} \text{ m}^3 \text{s}^{-2} \]
\[ c = 299792458 \text{ m s}^{-1} \]

6. Tropospheric correction

\[ Trop_{rec}^{sat} = \left( d_{dry} + d_{wet} \right) m(elev) = 6.76 \text{m} \]

\[ d_{dry} = 2.3 e^{-0.116 \times 10^{-3} H} = 2.3 \text{m} \]
\[ d_{wet} = 0.1 \text{m} \]
\[ m(elev) = \frac{1.001}{\sqrt{0.002001 + \sin^2(elev)}} \]

\[ \text{elev = 20.57} \frac{\pi}{180} = 0.359 \text{rad} \]
\[ H = 160 \text{m (height over the ellipsoid)} \]

See klob.f

\[ (x,y,z)_{rec} \xrightarrow{\text{[car2geo]}} (\text{Lon, Lat, H})_{rec} \]
7. Ionospheric correction

\[ (\text{time, } r_{\text{sta}}, r_{\text{sat}}, a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3) \rightarrow [\text{Klob}] \rightarrow \text{Iono}=10.26\text{m} \]

7. Compute the modeled pseudorange.

\[
C_l^{\text{sat, modelled}} = \rho_{\text{rec,0}} - c \left( \delta t^{\text{sat}} + \Delta r_{\text{sat}}^{\text{sat}} \right) + Trop_{\text{rec}}^{\text{sat}} + Ion_{\text{rec}}^{\text{sat}} + TGD_{\text{sat}}^{\text{sat}}
\]

\[
\rho_{\text{0,rec}} = 23616699.124\text{ m} \\
c \delta t^{\text{sat}} = 5.65 \cdot 10^{-6} \text{ c} = 1693.828\text{ m} \\
c \Delta r_{\text{sat}}^{\text{sat}} = -2.33 \cdot 10^{-10} \text{ c} = -0.071\text{ m} \\
Trop_{\text{rec}}^{\text{sat}} = 6.760\text{ m} \\
Ion_{\text{rec}}^{\text{sat}} = 10.260\text{ m} \\
TGD_{\text{sat}}^{\text{sat}} = -0.698\text{ m}
\]

\[ C_l^{\text{sat, modelled}} = 23615021.689\text{ m} \]
**Prefit residual:**

Is the difference between measured and modeled pseudorange

\[
\text{Pref}_{\text{rec}}^{\text{sat}} = C_1^{\text{sat}} - C_1^{[\text{mod}]^{\text{sat}}} = \rho_{\text{rec}}^{\text{sat}} - \rho_{0,\text{rec}}^{\text{sat}} + c\, dt_{\text{rec}} + K_{1\text{rec}} + \varepsilon
\]

In the previous example (PRN14 at \( t = 38230 \) s):

\[
\text{Pref} = 23585247.703 - 23615021.689 = -29773.986 \text{ m}
\]

Previously calculated

From measurement file

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Introduction: Linear model and Prefit-residual

Input:
- Pseudoranges (receiver-satellite j): $P_j$
- Navigation message. In particular:
  - satellite position when transmitting signal: $r = (x_l, y_l, z_l)$
  - offsets of satellite clocks: $dt_j$
    $(j = 1, 2, ..., n)$  $(n \geq 4)$

Unkowns:
- receiver position: $r = (x, y, z)$
- receiver clock offset: DT

For each satellite in view

\[ C_{1_{\text{rec}}}^{\text{sat}} = \rho_{\text{rec}}^{\text{sat}} + c \cdot (dt_{\text{rec}} - dt_{\text{sat}}) + \sum \delta_k + \varepsilon \]

Linearising $\rho$ around an ‘a priori’ receiver position $(x_{0,\text{rec}}, y_{0,\text{rec}}, z_{0,\text{rec}})$

\[
\begin{align*}
\rho_{\text{sat}}^{\text{sat}} & = \frac{x_{0,\text{rec}} - x_{\text{sat}}}{\rho_{0,\text{rec}}} \Delta x_{\text{rec}} + \frac{y_{0,\text{rec}} - y_{\text{sat}}}{\rho_{0,\text{rec}}} \Delta y_{\text{rec}} + \frac{z_{0,\text{rec}} - z_{\text{sat}}}{\rho_{0,\text{rec}}} \Delta z_{\text{rec}} + c\left(dt_{\text{rec}} - dt_{\text{sat}}\right) + \sum \delta_k \\
\end{align*}
\]

where:

\[
\begin{align*}
\Delta x_{\text{rec}} & = x_{\text{rec}} - x_{0,\text{rec}} ; \quad \Delta y_{\text{rec}} = y_{\text{rec}} - y_{0,\text{rec}} ; \quad \Delta z_{\text{rec}} = z_{\text{rec}} - z_{0,\text{rec}} 
\end{align*}
\]

Prefit-residuals (Prefit)

\[
\begin{align*}
C_{1_{\text{rec}}}^{\text{sat}} - \rho_{\text{rec}}^{\text{sat}} - c \cdot dt_{\text{sat}} - \sum \delta_k & = \frac{x_{0,\text{rec}} - x_{\text{sat}}}{\rho_{0,\text{rec}}} \Delta x_{\text{rec}} + \frac{y_{0,\text{rec}} - y_{\text{sat}}}{\rho_{0,\text{rec}}} \Delta y_{\text{rec}} + \frac{z_{0,\text{rec}} - z_{\text{sat}}}{\rho_{0,\text{rec}}} \Delta z_{\text{rec}} + c \cdot dt_{\text{rec}} \\
\end{align*}
\]
For all sat. in view

\[ \frac{\hat{T}_{sat n}}{\rho_{0,rec}} - \frac{\hat{T}_{sat n}}{\rho_{0,rec}} \]

Observations (measured-computed)

\( \hat{T}_{sat n} \)

Geometry of rays

Unitary Line-Of-Sight vector from receiver to satellite

\((x,y,z)\) coordinates

\( (e,n,u) \) coordinates

\[ \frac{x_{0,rec} - x_{sat}}{\rho_{0,rec}} \]

\[ \frac{y_{0,rec} - y_{sat}}{\rho_{0,rec}} \]

\[ \frac{z_{0,rec} - z_{sat}}{\rho_{0,rec}} \]

\[ \frac{x_{0,rec} - x_{sat}'}{\rho_{0,rec}'} \]

\[ \frac{y_{0,rec} - y_{sat}'}{\rho_{0,rec}'} \]

\[ \frac{z_{0,rec} - z_{sat}'}{\rho_{0,rec}'} \]

\[ \frac{x_{0,rec} - x_{sat}''}{\rho_{0,rec}''} \]

\[ \frac{y_{0,rec} - y_{sat}''}{\rho_{0,rec}''} \]

\[ \frac{z_{0,rec} - z_{sat}''}{\rho_{0,rec}''} \]

\[ \frac{x_{0,rec} - x_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{y_{0,rec} - y_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{z_{0,rec} - z_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{x_{0,rec} - x_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{y_{0,rec} - y_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{z_{0,rec} - z_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{x_{0,rec} - x_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{y_{0,rec} - y_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{z_{0,rec} - z_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{x_{0,rec} - x_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{y_{0,rec} - y_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{z_{0,rec} - z_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{x_{0,rec} - x_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{y_{0,rec} - y_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{z_{0,rec} - z_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{x_{0,rec} - x_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{y_{0,rec} - y_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{z_{0,rec} - z_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{x_{0,rec} - x_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{y_{0,rec} - y_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{z_{0,rec} - z_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{x_{0,rec} - x_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{y_{0,rec} - y_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{z_{0,rec} - z_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{x_{0,rec} - x_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{y_{0,rec} - y_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{z_{0,rec} - z_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{x_{0,rec} - x_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{y_{0,rec} - y_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{z_{0,rec} - z_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{x_{0,rec} - x_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{y_{0,rec} - y_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{z_{0,rec} - z_{sat}'''}{\rho_{0,rec}'''} \]

\[ \frac{x_{0,rec} - x_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{y_{0,rec} - y_{sat}''''}{\rho_{0,rec}''''} \]

\[ \frac{z_{0,rec} - z_{sat}''''}{\rho_{0,rec}''''} \]
From ECEF \((x,y,z)\) to Local \((e,n,u)\) coordinates

\[
\begin{bmatrix}
\Delta e \\
\Delta n \\
\Delta u
\end{bmatrix} = \mathbf{R}_1[^\pi/2 - \varphi] \mathbf{R}_3[^\pi/2 + \lambda] \begin{bmatrix}
\Delta x \\
\Delta y \\
\Delta z
\end{bmatrix}
\]

\[
\dot{\mathbf{e}} = (-\sin \lambda, \cos \lambda, 0)
\]

\[
\dot{\mathbf{u}} = (-\cos \lambda \sin \varphi, -\sin \lambda \sin \varphi, \cos \varphi)
\]

\[
\dot{\mathbf{u}} = (\cos \lambda \cos \varphi, \sin \lambda \cos \varphi, \sin \varphi)
\]

Of course, receiver coordinates \((x_{\text{rec}}, y_{\text{rec}}, z_{\text{rec}})\) are not known (they are the target of this problem). But, we can always assume that an “approximate position \((x_{0,\text{rec}}, y_{0,\text{rec}}, z_{0,\text{rec}})\) is known”.

Thence, the navigation problem will consist on:

1.- To start from an approximate value for receiver position \((x_{0,\text{rec}}, y_{0,\text{rec}}, z_{0,\text{rec}})\) e.g. the Earth’s centre to linearise the equations.

2.- With the pseudorange measurements and the navigation equations, compute the correction \((\Delta x_{\text{rec}}, \Delta y_{\text{rec}}, \Delta z_{\text{rec}})\) to have improved estimates:

\[
(x_{\text{rec}}, y_{\text{rec}}, z_{\text{rec}}) = (x_{0,\text{rec}}, y_{0,\text{rec}}, z_{0,\text{rec}}) + (\Delta x_{\text{rec}}, \Delta y_{\text{rec}}, \Delta z_{\text{rec}})
\]

3.- Linearise the equations again, about the new improved estimates, and iterate until the change in the solution estimates is sufficiently small.
For all sat. in view

\[
\begin{align*}
\text{Prefit}^1 & \left[ \begin{array}{c}
\text{Prefit}^2 \\
\text{..} \\
\text{Prefit}^n
\end{array} \right] \\
\end{align*}
\]

Observations (measured-computed)

Geometry of rays

\[
\begin{align*}
\Delta x_{\text{rec}} & \\
\Delta y_{\text{rec}} & \\
\Delta z_{\text{rec}} & \\
c & d t_{\text{rec}}
\end{align*}
\]

Thence, the basic linearized GPS measurement equation can be written as:

\[
y = G x
\]

This is a linear system with, in general, \( n \geq 4 \) equations which we can solve using LS, WLS, Kalman filter, ...

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Least Squares solution (conceptual review)

As a driving problem, let us consider the problem of fitting a set of points (noisy measurements) to a straight line \( y = mx + n \).

\[
\begin{array}{c|c}
 x & y \\
 x_1 & y_1 \\
 x_2 & y_2 \\
 \vdots & \vdots \\
 x_N & y_N \\
\end{array}
\]

This is an over-determined (incompatible) system of equations (due to the measurement noise \( \varepsilon \)).

It is evident that there is no straight line passing over all the data points (red points). Thence, we have to look for a solution that fits the measurements best in some sense.

Note that, as \( G \) is not an squared matrix \((N>2)\), we cannot try:

\[
y = Gp \Rightarrow p = G^{-1}y
\]

But, \( G^T G \) is a squared \((N \times N)\) matrix, thence, we can try:

\[
G^T y = G^T G p \Rightarrow \hat{p} = (G^T G)^{-1} G^T y
\]
Results from Linear Algebra:

1) \( \exists (G^T G)^{-1} \iff \text{The columns of matrix } G \text{ are linearly independents.} \)

2) \( \hat{p} = (G^T G)^{-1} G^T y \iff \min \|y - \hat{y}\| = \min \left( \sum_{i=1}^{N} (y_i - \hat{y}_i)^2 \right) \)

where \( \hat{y} = G \hat{p} \)

But, what is the physical meaning of the least square solution? What does it mean the condition

\[ \min \|y - \hat{y}\| = \min \left( \sum_{i=1}^{N} (y_i - \hat{y}_i)^2 \right) \]?
The Least Squares solution gives the solution of equilibrium of the following physical system, in which the red dots (i.e. data points) are attached to the straight line by springs that can only move in the direction of the y axis.

Indeed, the equilibrium solution is reached when the Total Potential Energy of the system is the minimum. That is, assuming the same spring constant \( k \):

\[
V_i = \frac{1}{2} k \Delta y_i^2 = \frac{1}{2} k (y_i - \hat{y}_i)^2
\]

\[
V_{TOTAL} = \frac{1}{2} k \sum_{i=1}^{N} (y_i - \hat{y}_i)^2
\]

\[
\min(V_{TOTAL}) \Rightarrow \min \left( \sum_{i=1}^{N} (y_i - \hat{y}_i)^2 \right)
\]

\[
\min \| \mathbf{y} - \hat{\mathbf{y}} \| = \min \left( \sum_{i=1}^{N} (y_i - \hat{y}_i)^2 \right) \quad \Leftrightarrow \quad \hat{\mathbf{p}} = (\mathbf{G}^T \mathbf{G})^{-1} \mathbf{G}^T \mathbf{y}
\]
Let be the basic linearized system:

\[ y = G \hat{x} \]

- Least Squares solution:
  \[ \hat{x} = (G^T G)^{-1} G^T y \]

The same error is assumed in all measurements.

- Weighted Least Squares solution

If the measurements have different errors, the equations can be weighted by matrix \( W \):

\[
\min \|y - \hat{y}\|^2_w = \min \left[ \sum w_i (y_i - \hat{y}_i)^2 \right]
\]

Uncorrelated errors are assumed.

And the weighted least squares solution is:

\[ \hat{x} = (G^T W G)^{-1} G^T W y \]

Weighted Least Squares solution:

The same, but assuming different spring constants \( w_i \):

\[
V_i = \frac{1}{2} w_i \Delta y_i^2 = \frac{1}{2} w_i (y_i - \hat{y}_i)^2
\]

\[
V_{TOTAL} = \frac{1}{2} \sum_{i=1}^{N} w_i (y_i - \hat{y}_i)^2
\]

\[
\min (V_{TOTAL}) \Rightarrow \min \left( \sum_{i=1}^{N} w_i (y_i - \hat{y}_i)^2 \right)
\]

\[
\min \|y - \hat{y}\|_W = \min \left( \sum_{i=1}^{N} w_i (y_i - \hat{y}_i)^2 \right) \Leftrightarrow \hat{p} = (G^T W G)^{-1} G^T W y
\]
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Let be the basic linearized C

\[ y = G \mathbf{x} \]

• Least Squares solution:
  \[ \hat{x} = (G'G)^{-1} G'y \]
  The same error is assumed in all measurements

• Weighted Least Squares solution

If the measurements have different errors, the equations can be weighted by matrix \( W \):

And the weighted least squares solution is:

\[ \hat{x} = (G'W G)^{-1} G'Wy \]

\[
\begin{bmatrix}
W_{y_1} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & W_{y_n}
\end{bmatrix}
\]

Uncorrelated errors are assumed

\[
\min \|y - \hat{y}\|_w = \min \sum_i w_i (y_i - \hat{y}_i)^2 \]

\[ \hat{y} = G \hat{x} \]
Assuming that measurements $Y$ have random errors with zero mean and variance $\sigma^2$, and assuming that error sources for each satellite are uncorrelated with error sources for any other satellite, the following weighted matrix may be used:

$$ W = \begin{bmatrix} 1/\sigma_{y_1}^2 & 0 \\ \vdots & \ddots \\ 0 & 1/\sigma_{y_n}^2 \end{bmatrix} $$

Best Linear Unbiased Minimum Variance Estimator (BLUE):

Let be $P_y$ the error covariance matrix for measurements $y$. If the weighting matrix is taken as $W = P_y^{-1}$, thence the Minimum Variance Solution is found:

$$ \hat{x} = \left( G^tP_y^{-1}G \right)^{-1} G^tP_y^{-1}y $$

And the error covariance matrix for the estimation $\hat{x}$ is:

$$ P_{\hat{x}} = \left( G^tP_y^{-1}G \right)^{-1} $$

---

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7. Navigation equations system and LS solution (XYZ)

Repeat the previous exercise, but writing the system and computing the solution in (XYZ) coordinates. Also, compute GDOP, Precision Dilution Of Precision (PDOP) and TDOP.

Complete the following steps:

(a) The matrix $\mathbf{G}$ is now

$$
\mathbf{G}_i = \begin{bmatrix}
    x_0 - x_i^i \\ y_0 - y_i^i \\ z_0 - z_i^i \\ \rho_0^i
\end{bmatrix}
$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$ is the ‘a priori’ receiver coordinates at reception time, $\mathbf{r}_i^i = (x_i^i, y_i^i, z_i^i)$ are the satellite coordinates at transmission time, and $\rho_0^i = ||\mathbf{r}_i^i - \mathbf{r}_0||$.

*Hint*: Matrix $\mathbf{G}$ and profit residual vector $\mathbf{y}$ can be generated directly from the gLAB.out output file as follows:

```bash
grep MODEL gLAB.out | grep C1 | gawk 'BEGIN{x=4789032.6277; y=176595.0498; z=4195013.2503} {if ($4==300 & & $6!=21) {r1=x-$11; r2=y-$12; r3=z-$13; r=sqrt(r1*r1+r2*r2+r3*r3); print $9-$10,r1,r2/r,r3/r,1}}' > M.dat
```

Vector $\mathbf{y}$ corresponds to the first column of file M.dat and matrix $\mathbf{G}$ to the last four columns.

See exercises 6 and 7, Session 5.2 in [RD-2]

(b) Compute the LS solution of the navigation system. Using Octave or MATLAB, upload the contents of file M.dat and execute the following instructions, as well:

```bash
y=M(:,1)
G=M(:,2:5)
x=inv(G'*G)*G'*y
```

The values computed by gLAB can be found by:

```
(X,Y,Z) coordinates:
grep OUTPUT gLAB.out | grep -v INFO |
gawk '{if ($4==300) print $9,$10,$11}'
```

Receiver clock:

```
grep FILTER gLAB.out | grep -v INFO |
gawk '{if ($4==300) print $8}'
```

The matrix $\mathbf{G}$ and vector $\mathbf{y}$ values computed by gLAB can be found by:

```bash
grep PREFIT gLAB.out | grep -v INFO |
gawk '{if ($4==300 & & $6!=21) print $8,$11,$12,$13,$14}’
```
Kalman filtering:

It is based on computing the weighted average between:

- the measurement \( y(n) \) (i.e., at \( t = t_n \))
- the prediction of the state \( \hat{x}^-(n) \) from previous estimation \( \hat{x}(n-1) \)

1. **Weighted average:**

\[
\begin{pmatrix}
  y(n) \\
  \hat{x}^-(n)
\end{pmatrix}
= 
\begin{pmatrix}
  G(n) \\
  \text{I}
\end{pmatrix}
\begin{pmatrix}
  x(n)
\end{pmatrix}
\]

Let’s assume, that we have the prediction \( \hat{x}^-(n) \), with \( \text{P}_{\hat{x}^-(n)} \), it can be used to add an additional set of equations to the measurement equation \( y = G \cdot x \)

\[
W = \begin{pmatrix}
  \text{P}_{y(n)} & 0 \\
  0 & \text{P}_{\hat{x}^-(n)}
\end{pmatrix}^{-1}
\]
\[
\begin{bmatrix}
\mathbf{y}(n) \\
\hat{\mathbf{x}}^{-}(n)
\end{bmatrix} = 
\begin{bmatrix}
\mathbf{G}(n) \\
\mathbf{I}
\end{bmatrix} \mathbf{x}(n)
\]

\[
\mathbf{W} = 
\begin{pmatrix}
\mathbf{P}_{\mathbf{y}(n)} & 0 \\
0 & \mathbf{P}_{\hat{\mathbf{x}}^{-}(n)}
\end{pmatrix}^{-1}
\]

And the following solution of the previous equation system can be found with some elemental algebraic manipulations:

\[
\hat{\mathbf{x}} = (\mathbf{G}'\mathbf{P}_{\mathbf{y}}^{-1}\mathbf{G})^{-1}\mathbf{G}'\mathbf{P}_{\mathbf{y}}^{-1}\mathbf{y}
\]

\[
\mathbf{P}_{\hat{\mathbf{x}}} = (\mathbf{G}'\mathbf{P}_{\mathbf{y}}^{-1}\mathbf{G})^{-1}
\]

\[
\hat{\mathbf{x}}(n) = \mathbf{P}_{\hat{\mathbf{x}}(n)} \left[ \mathbf{G}'(n) \mathbf{P}_{\mathbf{y}(n)}^{-1} \mathbf{y}(n) + \mathbf{P}_{\hat{\mathbf{x}}^{-}(n)}^{-1} \hat{\mathbf{x}}^{-}(n) \right]
\]

\[
\mathbf{P}_{\hat{\mathbf{x}}(n)} = \left[ \mathbf{G}'(n) \mathbf{P}_{\mathbf{y}(n)}^{-1} \mathbf{G}(n) + \mathbf{P}_{\hat{\mathbf{x}}^{-}(n)}^{-1} \right]^{-1}
\]

2.- Prediction

**Scalar case:**

Let’s \( \hat{\mathbf{x}}_{n-1} \) be the state at epoch \( n-1 \) with variance \( \sigma_{\hat{\mathbf{x}}_{n-1}}^2 \).

The **simplest prediction model** is to assume that the prediction at epoch \( n \) is proportional to the state at epoch \( n-1 \). That is:

\[
\hat{\mathbf{x}}_{n} = \phi \, \hat{\mathbf{x}}_{n-1}
\]

Thence, existing a linear relation between \( \hat{\mathbf{x}}_{n-1} \) and \( \hat{\mathbf{x}}_{n} \), the variance of the prediction will be:

\[
\sigma_{\hat{\mathbf{x}}_{n}}^2 = \phi^2 \, \sigma_{\hat{\mathbf{x}}_{n-1}}^2 + q^2
\]

An additional term is added to account for modeling error!
Generalization to the vector case:

\[ \hat{x}_n^- = \phi \hat{x}_{n-1} \]
\[ \sigma^2_{\hat{x}_n^-} = \phi^2 \sigma^2_{\hat{x}_{n-1}} + q^2 \]

\[ x_n \rightarrow \mathbf{x}(n) \]
\[ \phi \rightarrow \mathbf{\Phi}(n) \]
\[ \sigma^2_{x_n} \rightarrow \mathbf{P}_{\mathbf{x}(n)} \]
\[ q^2 \rightarrow \mathbf{Q}(n) \]

\[ \hat{x}^-(n) = \mathbf{\Phi}(n-1) \cdot \hat{x}(n-1) \]
\[ \mathbf{P}_{\hat{x}^-}(n) = \mathbf{\Phi}(n-1) \cdot \mathbf{P}_{\hat{x}(n-1)} \cdot \mathbf{\Phi}'(n-1) + \mathbf{Q}(n-1) \]

Kalman filter (see kalman.f)
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Some simple examples to define matrices $\Phi$ and $Q$

\[
\hat{x}^-(n) = \Phi(n-1) \cdot \hat{x}(n-1)
\]
\[
P_{\hat{x}^-}(n) = \Phi(n-1) \cdot P_{\hat{x}(n-1)} \cdot \Phi'(n-1) + Q(n-1)
\]

**a) Static positioning:**

State vector to be determined is $X = (\Delta x_{rec}, \Delta y_{rec}, \Delta z_{rec}, D_{t_{rec}})$ where coordinates $(\Delta x_{rec}, \Delta y_{rec}, \Delta z_{rec})$ are considered **constant** (because receiver is fixed) and clock offset $D_{t_{rec}}$ is treated as **white noise** with zero mean and variance $\sigma^2_{Dt}$. In these conditions, matrices have the form:

$$
\Phi(n) = \begin{bmatrix}
1 & 1 & 1 & 0
\end{bmatrix}
$$

$$
Q(n) = \begin{bmatrix}
0 & 0 & 0 & \sigma^2_{Dt}
\end{bmatrix}
$$

Being $\sigma^2_{Dt}$ process noise associated to clock offset (in some way, the uncertainty in clock value).

The coordinates are always the same!

**Constant**

\[
\Phi = 1 \iff x(n) = x(n-1)
\]

\[
Q = 0 \quad \text{(no prediction error)}
\]

We can assure that, the next $x(n)$ will be the same as $x(n-1)$. 

\[
\hat{x}^-(n) = \Phi(n-1) \cdot \hat{x}(n-1)
\]

\[
P_{\hat{x}^-}(n) = \Phi(n-1) \cdot P_{\hat{x}(n-1)} \cdot \Phi'(n-1) + Q(n-1)
\]
White Noise process $N(0, \sigma)$

From a given epoch is not possible to predict the following one

$\Phi=0 \leftrightarrow x(n)=0$ (zero mean)

$Q=\sigma^2$ (prediction error noise)

$x(n)=0 \text{ with a confidence } \sigma.$

$\hat{x}(n) = \Phi(n-1) \cdot \hat{x}(n-1)$

$P_{\hat{x}}(n) = \Phi(n-1) \cdot P_{\hat{x}}(n-1) \cdot \Phi^T(n-1) + Q(n-1)$

We only can assume that, the next $x(n)$ can be $x(n)=0$ with a confidence $\sigma$. 
b) Kinematic positioning

1) In case of a fast moving vehicle, coordinates will be modeled as white noise with zero mean, and the same rationale applies for clock offset:

\[
\Phi(n) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q(n) = \begin{pmatrix} \sigma_{dx}^2 & 0 \\ 0 & \sigma_{dy}^2 \end{pmatrix}
\]

2) In case of a slow moving vehicle, coordinates may be modeled as random walk, process’ spectral density \( \dot{q} = \frac{d\sigma^2}{dt} \), and the clock as a white noise:

\[
\Phi(n) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad Q(n) = \begin{pmatrix} \dot{q}_{dz} & \dot{q}_{dy} & \dot{q}_{dx} \\ \dot{q}_{dy} & \dot{q}_{dz} & 0 \\ \dot{q}_{dx} & 0 & \sigma_{DT}^2 \end{pmatrix}
\]
Random Walk process: it varies slowly

The uncertainty increases with time

\[ \frac{d\sigma^2}{dt} \Delta t \]

\[ \dot{x}(n) = \Phi(n-1) \cdot \dot{x}(n-1) \]
\[ P_k(n) = \Phi(n-1) \cdot P_{k(n-1)} \cdot \Phi'(n-1) + Q(n-1) \]

\( \Phi = 1 \leftrightarrow x(n) = x(n-1) \) (the same value is assumed)

\( Q = (d\sigma^2/\ dt)*\Delta t \) (but, with prediction error noise increasing with time)
Pure Kinematic positioning: white noise coordinates and clock

Kinematic positioning: random walk noise coordinates and white noise clock
Positioning: Random walk coordinates with $Q=0$ and white noise clock

Session 6a, exercise 5c.1: Deviations of the estimations referring to the nominal (random walk, $Q=0$)
8. Solving with the Kalman filter

The measurement file UPC11490.050 has been collected by a receiver with fixed coordinates. Using navigation file UPC11490.05N, compute the SPP solution in static mode and check by hand the computation of the solution for the first three epochs (i.e. $t = 300$, $t = 600$ and $t = 900$ seconds).

Complete the following steps:

(a) Set the default configuration of gLAB for the SPP mode. Then, in section [Filter], select [Static] in the Receiver Kinematics option. To process the data click [Run gLAB].

---

Solving with the kalman filter (by hand):
See exercise 8, Session 5.2 in [RD-2]

---

(c) Using the previous equations and the configuration parameters applied by gLAB compute by hand the solution for the first three epochs (i.e. $t = 300$, $t = 600$ and $t = 900$ s).

Note: Use the prefit residual vector $y(k)$ and design matrix $G(k)$ computed by gLAB.

Hint:

i. Filter configuration (according to gLAB):

- Initialisation:
  \[
  \bar{x}_0 = \bar{x}(0) = (0, 0, 0, 0),
  P_0 = P(0) = \sigma_0^2 I, \text{ with } \sigma_0 = 3 \cdot 10^5 \text{m}.
  \]

- Process noise $Q$ and transition matrices $\Phi$:
  \[
  Q \equiv Q(k) = \begin{bmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & \sigma_{dt}^2 & 0
  \end{bmatrix},
  \Phi \equiv \Phi(k) = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
  \end{bmatrix}
  \]
  with $\sigma_{dt} = 3 \cdot 10^5 \text{m}$.

- Measurement covariance matrix:
  \[
  R_k \equiv R(k) = \sigma_y^2 I, \text{ with } \sigma_y = 1 \text{m}.
  \]
(b) Write the Kalman filter equations according to Fig. 6.2, in section 6.1.2 of Volume 1.

\[
\begin{align*}
\hat{x}_{k+1}^- &= \Phi_k \hat{x}_k \\
\Sigma_{k+1}^- &= \Phi_k \Sigma_k \Phi_k^T + Q_k
\end{align*}
\]

\[
\hat{x}_k = P_k \left[ G_k^T R_k \Sigma_k + (P_k^-)^{-1} \hat{x}_k^- \right]^{-1}
\]

\[
P_k = \left[ G_k^T R_k \Sigma_k + (P_k^-)^{-1} \right]^{-1}
\]

\[
\hat{x}_0, P_0
\]

\[
\hat{x}_k, P_k
\]

---

ii. Kalman filter iterations:

k=1:

Predict:

\[
x_1^- = \hat{\phi} \cdot \hat{x}_0
\]

\[
P_1^- = \hat{\phi} \cdot P_0 \cdot \hat{\phi}^T + Q
\]

Update:

\[
P_1 = \left[ G_1^T \cdot R_1 \Sigma_1 + (P_1^-)^{-1} \right]^{-1}
\]

\[
\hat{x}_1 = P_1 \cdot \left[ G_1^T \cdot R_1 \Sigma_1 \cdot y_1 + (P_1^-)^{-1} \cdot x_1^- \right]
\]

k=2:

Predict:

\[
x_2^- = \hat{\phi} \cdot \hat{x}_1
\]

\[
P_2^- = \hat{\phi} \cdot P_1 \cdot \hat{\phi}^T + Q
\]

Update:

\[
P_2 = \left[ G_2^T \cdot R_2 \Sigma_2 + (P_2^-)^{-1} \right]^{-1}
\]

\[
\hat{x}_2 = P_2 \cdot \left[ G_2^T \cdot R_2 \Sigma_2 \cdot y_2 + (P_2^-)^{-1} \cdot x_2^- \right]
\]

k=3:

\[
\vdots
\]

iii. Data vectors and matrices: Vectors \( y_k \equiv y(k) \) and design matrices \( \Sigma_k \equiv G(k) \) are generated from the \texttt{gLAB.out} file.
Execute for instance:\(^7^6\)

```
grep "PREFIT" gLAB.out | grep -v INFO |
gawk '{if ($6!=21)print $0}|
gawk '{if ($4==300) print $8,$11,$12,$13,$14}'
> M300.dat

grep "PREFIT" gLAB.out | grep -v INFO |
gawk '{if ($4==600) print $8,$11,$12,$13,$14}'
> M600.dat

grep "PREFIT" gLAB.out | grep -v INFO |
gawk '{if ($4==900) print $8,$11,$12,$13,$14}'
> M900.dat
```

Then using Octave or MATLAB:

```
y1=M300(:,1)
G1=M300(:,2:5)
y2=M600(:,1)
G2=M600(:,2:5)
y3=M900(:,1)
G3=M900(:,2:5)
```

iv. Results computed by gLAB:

A. (X,Y,Z) coordinates:

```
grep OUTPUT gLAB.out | grep -v INFO |
gawk '{if ($4==300) print $9,$10,$11}'

grep OUTPUT gLAB.out | grep -v INFO |
gawk '{if ($4==600) print $9,$10,$11}'

grep OUTPUT gLAB.out | grep -v INFO |
gawk '{if ($4==900) print $9,$10,$11}'
```

B. Receiver clock

```
grep FILTER gLAB.out | grep -v INFO |
gawk '{if ($4==300) print $8}'

grep FILTER gLAB.out | grep -v INFO |
gawk '{if ($4==600) print $8}'

grep FILTER gLAB.out | grep -v INFO |
gawk '{if ($4==900) print $8}'
```
References


Thank you
Lecture 4
Introduction to DGNSS

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5 March 2017
Contents

1. Introduction: GNSS positioning and measurement errors.
4. DGNSS implementations: RTK, LADGNSS, WADGNSS.
5. DGNSS commercial services.
**GNSS Positioning**

**Standalone Positioning:** GNSS receiver autonomous positioning using broadcast orbits and clocks (SPS, PPS).

**Differential Positioning:** GNSS augmented with data (differential corrections or measurements) from a single reference station or a reference station network.

Errors are similar for users separated tens, even hundred of kilometres, and these errors are removed/mitigated in differential mode, improving positioning.
Errors on the Signal

- Space Segment Errors:
  - Clock errors
  - Ephemeris errors

- Propagation Errors
  - Ionospheric delay
  - Tropospheric delay

- Local Errors
  - Multipath
  - Receiver noise

Selective Availability (S/A) was an intentional degradation of public GPS signals implemented for US national security reasons.

S/A was turned off at May 2\textsuperscript{nd} 2000 (Day-Of-Year 123).

It was permanently removed in 2008, and not included in the next generations of GPS satellites.

In the 1990s, the S/A motivated the development of DGPS.
- These systems typically computed PseudoRange Corrections (PRC) and Range-Rate Corrections (RRC) every 5-10 seconds.
- With S/A=off the life of the corrections was increased to more than one minute.
Contents

1. Introduction: GNSS positioning and measurement errors.
4. DGNSS implementations: RTK, LADGNSS, WADGNSS.
5. DGNSS commercial services.
Most of the errors cancel out when computing the difference between “BELL” and “EBRE” solutions. (the same satellites are used in both solutions)

The determination of the vector between the receivers APCs (i.e. the baseline “b”) is more accurate than the single receiver solution, because common errors cancel
The determination of the vector between the receivers APCs (i.e. the baseline “b”) is more accurate than the single receiver solution, because common errors cancel.

If the coordinates of the reference receiver are known, thence the reference receiver can estimate its positioning error, which can be transmitted to the user. Then, the user can apply these corrections to improve the positioning.

Note: Actually the corrections are computed in range domain (i.e. for each satellite) instead of in the position domain.
In the previous example, the differential error has been cancelled in the “position” domain (*i.e. solution domain approach*).

**But:**

**It requires to use the same satellites in both stations.**

Thence, is much better to solve the problem in the “range” domain than in the “position” domain. That is, to provide corrections for each satellite in view (*i.e. measurement domain approach*).

Two implementations can be considered:

1.- The reference station, with known coordinates, computes range corrections for each satellite in view. These corrections are broadcasted to the user. The user applies these corrections to compute its “absolute position”.

2.- The reference receiver (not necessarily at rest) broadcasts its time-tagged measurements to the user. The user applies these measurements to compute its “relative position” to the reference station. Note: if the reference station coordinates are known, the user can estimate its absolute position, as well.

---

**1.- Range Differential Correction Calculation**

- The **reference station** with known coordinates, computes pseudorange and range-rate corrections: $P_{RC} = \rho_{ref} - \rho_{ref}$, $R_{RC} = \Delta P_{RC}/\Delta t$.

- The **user** receiver applies the PRC and RRC to correct its own measurements, $P_{user} + (P_{RC} + R_{RC}(t-t_0))$, removing SIS errors and improving the positioning accuracy.

*DGNSS with code ranges*: users within a hundred of kilometres can obtain **one-meter-level** positioning accuracy using such pseudorange corrections.
Differential Positioning Performance

GPS Standalone

S/A=off

DGPS
In the previous example, the differential error has been cancelled in the “position” domain (i.e. solution domain approach).

But:

It requires to use the same satellites in both stations.

Thence, is much better to solve the problem in the “range” domain than in the “position” domain. That is, to provide corrections for each satellite in view (i.e. measurement domain approach).

Two implementations can be considered:

1.- The reference station, with known coordinates, computes range corrections for each satellite in view. These corrections are broadcasted to the user. The user applies these corrections to compute its “absolute position”.

2.- The reference receiver (not necessarily at rest) broadcast its time-tagged measurements to the user. The user applies these measurements to compute its “relative position” to the reference station. Note: if the reference station coordinates are known, the user can estimate its absolute position, as well.
2.- Differential GNSS (DGNSS): Relative position

This concept of DGNSS can be applied even if the position of the reference station is not known accurately or is moving, as well. In this case, the user estimates its relative position vector with the reference receiver.

In this implementation of DGPS, the reference station broadcast its time-tagged measurements rather than the computed differential corrections. The user receiver form differences of its own measurements with those at the reference receiver, (satellite by satellite) and estimate its position relative to the reference receiver.

Real-Time Kinematics (RTK) is and example of this DGNSS. Users within some ten of kilometres can obtain centimetre level positioning. The baseline is limited by the differential ionospheric error that can reach up to 10cm, or more, in 10km, depending of the ionospheric activity.
COMMENTS

Real-Time implementation entails delays in data transmission, which can reach up to 1 or 2 s.

- Differential corrections vary slowly and its useful life is of several minutes (S/A=off)
- But, the measurements change much faster:
  - The range rate $\frac{d\rho}{dt}$ can be up to 800 m/s and, therefore, the range can change by more than half a meter in 1 millisecond. Moreover the receiver clock offset can be up to 1 millisecond (depending on the receiver configuration).

- Thence, the reference station measurements must be:
  - Synchronized to reduce station clock mismatch: station clock can be estimated to within $1 \mu$s $\epsilon_{dt_{sta}} < 1$ mm
  - Extrapolated to reduce error due to latency: carrier can be extrapolated with error < 1 cm.

\[ RRC = \frac{\Delta PRC}{\Delta t} \]

\[ dL_1 \approx d\rho + dT_{rec} \]

$\frac{dL_1}{dt}$ up to $\sim 800$ m/s
$\frac{dT_{rec}}{dt}$ up to $\sim 670$ m/s

Receiver: JAVAD TRE_G3TH DELTA3.12

---

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Master of Science in GNSS
1. Introduction: GNSS positioning and measurement errors.
4. DGNSS implementations: RTK, LADGNSS, WADGNSS.
5. DGNSS commercial services.
Error mitigation: DGNSS residual error

Errors are similar for users separated tens, even hundred of kilometres, and these errors vary ‘slowly’ with time. That is, the errors are correlated on space and time.

The spatial decorrelation depends on the error component (e.g. clocks are common, ionosphere ~100km...). Thence, a reference stations network is needed to cover a wide-area.

Space Segment Errors

- **Satellite clock error:**
  - Clock modelling error is small (~2m RMS) and changes slowly over hours.
  - Does not depend on user location, thence, it can be eliminated in differential mode.

- **Satellite ephemeris:**
  - Only the Line-Of-Sight (LOS) of error affects positioning. This error is small (~2m RMS) and changes slowly over minutes.
  - The residual error, after applying the differential corrections depends upon the separation between the LOS from user and reference station.
  - A conservative bound is given by:
    \[
    \delta \rho < \frac{b}{\rho} \delta e
    \]
    
    with a baseline \( b = 20km \)
    
    \[
    \delta \rho < \frac{20km}{20000km} \cdot \frac{1}{1000} \delta e
    \]
Ephemeris Errors and Geographic decorrelation

Differential range error due to satellite orbit error

\[
\delta \rho = \frac{\mathbf{e} \cdot \mathbf{p}_\text{user} - \mathbf{e} \cdot \mathbf{p}_\text{ref}}{\mathbf{p}_\text{user}} - \frac{\mathbf{p}_\text{ref}}{\mathbf{p}_\text{ref}}
\]

\[
= - \mathbf{e}^T \left(1 - \mathbf{p}^T \mathbf{p}^T\right) \frac{b}{\rho}
\]

A conservative bound:

\[
\delta \rho < \frac{b}{\rho} \epsilon
\]

with a baseline \(b = 20\, \text{km}\)

\[
\delta \rho < \frac{20}{20000} \epsilon = \frac{1}{1000} \epsilon
\]
Satellite location error

Range error from CREU and EBRE

Differential range error from between CREU and EBRE
288 km of baseline

\[ \delta \rho = \vec{e} \cdot \frac{\vec{p}_{\text{user}}}{\rho_{\text{user}}} - \vec{e} \cdot \frac{\vec{p}_{\text{ref}}}{\rho_{\text{ref}}} \]

CREU-EBRE: 288 km; SPP: 2000 m Along-Track orbit error

288 km of baseline

Absolute positioning
Satellite clock anomaly

Reference stations can detect and remove clock failures and other anomalies (e.g. GBAS)

Atmosphere Propagation Errors

- Ionospheric propagation delay:
  - Ionospheric delay depends on the STEC (integrated electron density along ray path).
  - Reference and user receiver locations (i.e. Baselines) are mapped to Ionospheric Pierce Points (IPPs) associated to each satellite.
  - Typical spatial gradients of ionosphere are 1-2 mm/km (1σ) ➔ 0.1-0.2 m in 100 km. This value can reach up to **300 mm/km** (6-7 April 2000)
Atmosphere Propagation Errors

- **Tropospheric propagation delay:**
  - Tropospheric delay depends upon the air density profile along the signal path.
  - Most of the tropospheric delay (~90%) comes from the predictable hydrostatic component.
  - Wet component delay can vary considerably, both spatially and temporally.
  - With 10 km separation between receivers, the residual range error can be 0.1-0.2 m.
  - For long distance or significant altitude difference it is preferable to correct for the tropospheric delay at both reference and user receivers. For a low elevation satellite, the residual range error can be 2-7 mm per meter of altitude difference.
Zenith Tropospheric delay: from DoY 052-059 Year 2013

Nominal

Actual

Zenith Tropospheric Delay geographical decorrelation

Nominal

Actual

Zenith Wet Tropospheric Geographical decorrelation

Vertical delay

Vertical delay
Local Errors

- **Receiver noise and multipath:**
  - These errors are uncorrelated at the reference and user receivers and cannot be corrected by DGPS.
  - In fact, any error incurred in the reference station affects the user. Thence, it is important to minimize errors at the reference station.
  - Code noise can be reduced by smoothing with carrier (at the level of 0.25-0.50m). But single frequency smoothing is affected by code-carrier ionosphere divergence.
  - High accuracy applications use carrier measurements, about two orders of magnitude more precise than code measurements, but the unknown ambiguities must be fixed.

---

**Ionospheric delay (STEC)**

Halloween storm
Error in carrier measurement due to multipath (cm level) or thermal noise (mm level) is typically 2 orders of magnitude lower than in code, but carrier has an unknown ambiguity.

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**ERRORS on the Signal**

- **Space Segment Errors:**
  - Clock errors
  - Ephemeris errors

- **Propagation Errors**
  - Ionospheric delay
  - Tropospheric delay

- **Local Errors**
  - Multipath
  - Receiver noise

**Strong spatial correlation**

**Weak spatial correlation**

**No spatial correlation**
## Contents

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---

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<th>Source</th>
<th>Potential Error size</th>
<th>Error mitigation &amp; Residual error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Satellite clock</td>
<td>Clock modelling error: 2 m (RMS)</td>
<td>DGPS: 0.0m</td>
</tr>
<tr>
<td>Ephemeris prediction</td>
<td>Line-Of-Sight error: 2 m (RMS)</td>
<td>DGPS: 0.01m (RMS)</td>
</tr>
<tr>
<td>Ionospheric Delay</td>
<td>Vertical delay: ~ 2-10 m (depending upon user location, time of day &amp; solar activity)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Obliquity factor: 1 at zenith, 1.8 at 30°, 3 at 5°.</td>
<td>Single-freq. using Klobuchar: 1-5m.</td>
</tr>
<tr>
<td>Tropospheric Delay</td>
<td>Vertical delay ~ 2.3-2.5 m at sea level. (lower at a higher altitudes)</td>
<td>Model based on average meteorolog. Conditions: 0.1 -1 m DGPS: 0.2m (RMS) plus altitude effect.</td>
</tr>
<tr>
<td></td>
<td>Obliquity factor: 1 at zenith, 2 at 30°, 4 at 15° and 10 at 5°.</td>
<td></td>
</tr>
<tr>
<td>Multipath</td>
<td>In clean environment: Code: 0.5 – 1 m Carrier: 0.5 -1 cm</td>
<td>Uncorrelated between antennas. Mitigation trough antenna design and sitting and carrier smoothing of code.</td>
</tr>
<tr>
<td>Receiver noise</td>
<td>Code: 0.25 – 0.50m (RMS) Carrier: 1-2 mm (RMS)</td>
<td>Uncorrelated between receivers.</td>
</tr>
</tbody>
</table>


DGPS is based assuming baselines of tens of km and signal latency of tens of seconds.
Differential GNSS (DGNSS) IMPLEMENTATIONS

Differential Positioning: GNSS augmented with data (differential corrections or measurements) from a single reference station or a reference station network.

The differential corrections can be broadcast as an:

- **“Scalar” correction (Local Area)**, where all corrections are lumped together.
  Examples: GBAS, LAAS, RASAN, RTK, VRS.

- **“Vector” correction (Wide Area)**, where the corrections are given for each error source separately (state-space approach)
  Examples: SBAS (WAAS, EGNOS...), DGPS (JPL), Fast-PPP.

Real Time Kinematics (RTK)

In this implementation of DGPS, the reference station broadcast its time-tagged measurements rather than the computed differential corrections. The user receiver form differences of its own measurements with those at the reference receiver, satellite by satellite and estimate its position relative to the reference receiver.

- Real-Time-Kinematics (RTK) is a technique for relative precise positioning based on carrier phase data. The key feature of RTK is the ability to fix the carrier ambiguities On-The-Flight (OTF), i.e. while on the move. Major receivers manufacturers offer RTK solution packages consisting on a pair or receivers, a radio link, and software.

  The performance of RTK is measured by (i) initialization time, and (ii) reliability (or, correctness) of the ambiguity fixing. There is an obvious trade-off between getting the answer quickly and getting it right.

  For typical baselines of several kilometres, integer ambiguity resolution in thirty or sixty seconds is common, achieving centimetre error level of accuracy.
Note: In Carrier phase Differential positioning, with a single reference station, the accuracy decreases as a function of the distance from the reference station by a rate of about 0.5-1cm per km.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Horizontal accuracy</th>
<th>Typical baselines up to 10-15km, depending on the ionosphere conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>static</td>
<td>5 mm + 0.5 ppm</td>
<td></td>
</tr>
<tr>
<td>Kinematic</td>
<td>5 cm + 5 ppm</td>
<td></td>
</tr>
</tbody>
</table>

Message Type | Title                                                                 |
-------------|-----------------------------------------------------------------------|
1            | Differential GPS corrections                                          |
2            | Delta Differential corrections                                        |
3            | GPS reference station parameters                                      |
9            | GPS partial satellite set                                             |
10           | P-code differential corrections                                       |
11           | GPS C/A code L1, L2 delta corrections                                 |
15           | Ionospheric delay message                                             |
17           | GPS ephemeris                                                         |
18           | RTK uncorrected carrier phases                                        |
19           | RTK uncorrected code pseudoranges                                    |
20           | RTK carrier phase corrections                                         |
21           | RTK code pseudorange corrections                                      |
59           | Proprietary message                                                  |

The NTRIP protocol has been defined For transmission RTCM data over internet
Local Area DGNSS (LADGNSS): VRS

LADGNSS includes a Master station and several monitor stations. The master station collects the range measurements of the monitor stations and process the data to generate the range corrections, which are broadcasted to users.

- An example for High Accuracy Positioning, is the Virtual Reference Station (VRS) technique, which is based on generating measurements of a virtual (non-existing) reference station, close to the user, from real measurements of a multiple reference station network. These virtual measurements are transmitted to the user to compute the RTK solution. The NRTK yields accuracies at the level of 5cm for baselines up to 40km.
Virtual Reference Station

The basic scenario for VRS surveying is as follows:

• The user goes to a work site. The work site preferably is, as with any other style of GPS surveying, physically surrounded by the reference station network.

• The user sets up a surveying job and logs into the Real-Time Network (RTN) system using a cell phone (or other communication method). As part of the login process, the rover sends its position (via a NMEA string) to the RTN system.

• The RTN system computes a virtual reference station, in close proximity to the rover based on the position sent. Using input from the closest surrounding reference stations, the RTN system then computes and sends corrections as if a real base station were broadcasting from the location of the virtual reference station.

• Using the cell phone, the receiver then obtains and applies the corrections in real time.

After initialization, the survey proceeds in exactly the same manner as an RTK survey.

http://water.usgs.gov/osw/gps/real-time_network.html

Benefits of VRS surveying over traditional RTK surveying include:

• No need for a user to establish a permanent/semi-permanent base station. This eliminates the time for initial site selection and (daily) set-up, any issues of security and power supply, and the possibility of set-up errors.

• The RTN can monitor its own integrity and can detect if there is a problem with a particular reference station. With a single-base RTK setup it can be difficult to tell if a problem exists or occurs with a base station while conducting a survey.

• Since the reference stations are part of a network, a loss of one station does not result in failure of the entire network or the resulting survey. Whereas the loss of a reference station with single-base RTK setup results in the end of data collection, with RTN surveying the system accuracy degrades gradually.

• A sufficiently dense reference station network can result in shorter baselines. As with any other style of GPS surveying, shorter baselines result in improved accuracy because of reduced effects of atmospheric interference.

• The RTN reference stations allow for network atmospheric modeling resulting in improved accuracy. With RTK, atmospheric effects are computed using (usually) one location.

• All users of the system are using a common, established ref. coordinate frame.
Limitations of VRS surveying include:

- There is a high cost of setting up and maintaining the RTN and to use an RTN setup by other organizations there is typically a yearly subscription fee that must be paid for network access.

- Use of the RTN can be limited by cell phone coverage and system down times.

- Availability is dependent on network extent and accuracy can be affected by the network density.

http://water.usgs.gov/ows/gps/real-time_network.html

Comment:
VRS is the most widely used implementation method of Network RTK (NRTK). Other possible implementations are briefly described in:

http://www.wasoft.de/e/iagwg451/intro/introduction.html

Local Area DGNSS (LADGNSS): GBAS

LADGNSS includes a Master station and several monitor stations. The master station collects the range measurements of the monitor stations and process the data to generate the range corrections, which are broadcasted to users.

- Examples using L1 carrier smoothed code are the Local Area Augmentation System (LAAS) or the Ground Based Augmentation System (GBAS), where a ground facility computes differential corrections and integrity data from measurements collected by several redundant receivers. This system is used to support aircraft operations during approach and landing. The differential corrections are transmitted on a VHF channel, up to about 40km. Meter level accuracies with integrity fulfilling the stringent requirements of Civil Aviation are met.
Differential GNSS (DGNSS) IMPLEMENTATIONS

**Differential Positioning**: GNSS augmented with data (differential corrections or measurements) from a single reference station or a reference station network.

The differential corrections can be broadcast as an:

- **“Scalar” correction (Local Area)**, where all corrections are lumped together.
  Examples: GBAS, LAAS, RASAN, RTK, VRS.

- **“Vector” correction (Wide Area)**, where the corrections are given for each error source separately (*state-space approach*).
  Examples: SBAS (WAAS, EGNOS...), DGPS (JPL), Fast-PPP.
Wide Area DGNSS (WADGNSS)

Differential Corrections to cover Continent-wide (or world-wide) users must be broadcast for each error source separately: Satellite clocks, ephemeris and ionosphere.

These corrections are computed by a Central processing Facility (CPF) from the range measurements of the monitor stations network with baselines of several hundreds up to thousand of kilometres.

- Examples using L1 carrier smoothed code are the Satellite Based Augmentation Systems (SBAS), e.g. WAAS, EGNOS, MSASS, for Civil Aviation, where differential corrections and integrity data fulfilling the Civil aviation requirements are broadcast over continental areas by a GEO satellite. Meter level accuracies with integrity are met.

Evolution to a dual frequency (L1,L5) system.

Error Mitigation

<table>
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<tr>
<th>Error component</th>
<th>Local Area (GBAS)</th>
<th>Wide Area (SBAS)</th>
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<tr>
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<tr>
<td>Multipath and Receiver Noise</td>
<td>Carrier Smoothing by user</td>
<td></td>
</tr>
</tbody>
</table>
For dual frequency users, the JPL-NASA provides Real-Time Global Differential GPS (GDGPS) to worldwide users to achieve decimetre/centimetre level of accuracy, after the best part of one hour, using the Precise Point Positioning (PPP) technique.

Indeed, precise orbits and clocks are computed from a global sparse reference stations network. The ionospheric error is eliminated from the combination of the two frequencies.
Current research: The Fast-PPP

Fast-PPP have prove that the high accuracy positioning of PPP users, can be reached very quickly (almost instantaneously) over continental areas with the 3-frequency Galileo signals. Moreover:

- World-wide users can perform **undifferenced ambiguity fixing**, thanks to the broadcasting of the fractional part of carrier ambiguities.
- Continent-wide area users (e.g. Europe) can achieve the accuracy quickly (few minutes, with **2-frequency** or single epoch with **3-frequency**, instead the 1/2-1 hour of PPP), thanks to the broadcasting of highly accurate ionospheric corrections.
- **Single frequency** users can take also benefit of the highly accurate ionospheric corrections achieving submetre positioning since the beginning and decimetre level positioning after the best part of one hour.

As in any WADGNSS, the differential corrections (satellite clocks, ephemeris, ionosphere, fractional part of ambiguities and DCBs) are computed by CPF from the measurements collected by an sparse network (few tens world wide distributed plus 30-40 for the continental enhancement of ionosphere model)

The Fast-PPP technique has been developed gAGE/UPC and is protected by several ESA-funded patents.

---

### CPF Corrections

<table>
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<tr>
<th>Correction</th>
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<td>Global</td>
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<td>Orbit Corrections</td>
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<tr>
<td><strong>Ionospheric</strong></td>
<td>Continental</td>
<td>300s</td>
<td>Iono. corrections</td>
<td>+ Fast PPP + Single Frequency</td>
</tr>
<tr>
<td><strong>Integrity</strong></td>
<td>Global/Continental</td>
<td>~5/300s</td>
<td>Confidence bounds</td>
<td>+ Integrity</td>
</tr>
</tbody>
</table>

The additional required bandwidth of F-PPP is about 10% of the classical PPP bandwidth.
MLVL (violet), EUSK (light blue) and EIJS (brown), at 252, 170 & 94 km respectively far from the nearest reference receiver BRUS.

Contents

1. Introduction: GNSS positioning and measurement errors.
4. DGNSS implementations: RTK, LADGNSS, WADGNSS.
5. DGNSS commercial services.
DGNSS Commercial services

Commercial WADGNSS services are already operational and in world-wide use for different applications: agriculture (e.g. OmniSTAR or CenterPoint RTX from Trimble), operations at sea (e.g. Starfix and Skyfix from Fugro), among others


• **OmniSTAR** provides four levels of service: [http://www.omnistar.com/](http://www.omnistar.com/)
  - Virtual base Station (VBS) offering sub-meter positioning,
  - World-wide service “XP” delivering better than 20 centimeter accuracy,
  - High performance (HP) service delivering greater than 10 centimeter accuracy
  - OmniSTAR “G2” service combines GPS plus GLONASS-based corrections to provide decimeter level positioning.

  OmniSTAR services were initially introduced by Fugro company and in 2011 was acquired by Trimble company

• Similar levels of services are provided by **Starfix**: [http://www.starfix.com](http://www.starfix.com)
  - Starfix.L1, Starfix.XP, Starfix.HP, Starfix.G2

---

**OmniSTAR HP**

10 cm High Performance

SubscriptionServices/OmniSTARHP.aspx

**OmniSTAR HP (10cm)** service is the most accurate solution available in the OmniSTAR portfolio of correction solutions. It is a L1/L2 solution requiring a dual frequency receiver.

OmniSTAR HP corrections are modeled on a network of reference sites using carrier phase measurement to maximize accuracy.

The expected 2-sigma (95%) accuracy of OmniSTAR HP is 10cm. It is particularly useful for Agricultural Machine guidance and many surveying tasks. It operates in real time and without the need for local Base Stations or telemetry links. OmniSTAR HP is a true advance in the use of GPS for on-the-go precise positioning.
OmniSTAR XP: 15 cm Worldwide Service

OmniSTAR XP (15cm) is a **worldwide dual frequency** high accuracy solution. It is a L1/L2 solution requiring a dual frequency receiver.

Orbit and Clock correction data is used together with atmospheric corrections derived from the dual frequency data.

By utilizing carrier phase measurement, very high accuracy can be achieved. OmniSTAR XP service provides short term accuracy of 1-2 inches and long term repeatability of better than 10 centimeters, 95% CEP.

It is especially suited for Agricultural automatic steering systems. While it is slightly less accurate than OmniSTAR HP, it is available worldwide and its accuracy is a significant improvement over regional DGNSS such as WAAS.

---

OmniSTAR VBS is the foundational **"sub-meter"** level of service. It is an **L1 only**, code phase pseudo-range solution.

Pseudo-range correction data from OmniSTAR’s regional reference sites is broadcast via satellite link to the user receiver.

These data are used, together with atmospheric modeling and knowledge of the receiver’s location, to generate an internal RTCM SC104 correction specific to that location. This correction is then applied to the R-T solution.

A typical 24-hour sample of OmniSTAR VBS will show a 2-sigma (95%) of significantly less than 1 meter horizontal position error and the 3-sigma (99%) horizontal error will be close to 1 meter.
OmniSTAR G2 is a **worldwide dual frequency** high-accuracy solution which uses Orbit and Clock correction data.

OmniSTAR G2 includes GLONASS satellites and GLONASS correction data in the solution. The addition of GLONASS to the solution **increases the number of satellites available** which is useful when faced with conditions that limit satellite visibility, such as terrain, vegetation or buildings.

OmniSTAR G2 service provides short-term accuracy of 1-2 inches and long term repeatability of better than 10 cm, 95% CEP. It is especially suited for operations in areas where trees or buildings may block the view of the sky and in areas affected by scintillation during times of high sunspot activity.

<table>
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<th>Baseline</th>
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<th>Init. time</th>
<th>Examples of Products</th>
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<td>Smoothed code (L1)</td>
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<td>PRC, RRC</td>
<td>~ metre</td>
<td>Single epoch</td>
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<td>RTK</td>
<td>Carrier L1/L2, L1</td>
<td>&lt;10-15km</td>
<td>Carrier measurements</td>
<td>~ cm</td>
<td>~10 s</td>
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<tr>
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<td>Smoothed code (L1)</td>
<td>&lt; 40 km</td>
<td>PRC, RRC</td>
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<td>Single epoch</td>
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<tr>
<td>VRS</td>
<td>Carrier L1/L2, L1</td>
<td>&lt; 50 km</td>
<td>Virtual Carrier measurements</td>
<td>few cm</td>
<td>~10 - 30 s</td>
</tr>
<tr>
<td>SBAS</td>
<td>Smoothed code (L1)</td>
<td>Continental</td>
<td>Orbits+ Clocks+ Ionosphere</td>
<td>~ metre + Integrity</td>
<td>Single epoch</td>
</tr>
<tr>
<td>GDGNSS</td>
<td>Iono-free code and carrier</td>
<td>Worldwide</td>
<td>Orbits + Clocks</td>
<td>~ dm</td>
<td>1/2h-1h.</td>
</tr>
<tr>
<td>F-PPP</td>
<td>L1, L2 code and carrier</td>
<td>Worldwide with Continental enhancement</td>
<td>Orbits+ Clocks+ Ionosphere+ DCBs + Frac. Ambig.</td>
<td>~ dm/cm (ambiguity fixing capability)</td>
<td>~5 m</td>
</tr>
</tbody>
</table>


Thank you
Backup slides

Example of Differential Atmospheric propagation effects analysis

USN3  23.6 km  GODN

GODS  76 m

DGPS

Vertical Error: 2013/02/21

DGPS

Horizontal Error: 2013/02/21
Single and double differences of receivers/satellites

\[ \nabla \mathbf{d} = \mathbf{k} - \mathbf{R} \]

Receiver errors affecting both satellites are removed (e.g. Receiver clock)

\[ \Delta \mathbf{d} \equiv \Delta \mathbf{d}_{\text{rov}} - \Delta \mathbf{d}_{\text{ref}} \]

SIS errors affecting both receivers are removed (e.g. Satellite clocks,...)

\[ \Delta \nabla \mathbf{d} = \Delta \mathbf{d}_{\text{rov}} - \Delta \mathbf{d}_{\text{ref}} \]

Receiver errors common for all satellites do not affect positioning (as they are assimilated in the receiver clock estimate). Thence:

- Only residual errors in single differences between sat. affect absolute positioning.
- Only residual errors in double differences between sat. and receivers affect relative positioning.

**Exercise**

Discuss the previous sentences.

Depicting atmosphere propagation errors affecting DGNSS: Double-differences between satellites and receivers

\[ L_{\text{sat}}^{\text{rec}} = \rho_{\text{rec}}^{\text{sat}} + c \cdot (dt_{\text{rec}} - dt_{\text{sat}}) + T_{\text{rec}}^{\text{sat}} - I_{\text{rec}}^{\text{sat}} + \lambda_{1} \Delta \omega_{\text{rec}}^{\text{sat}} + b_{1, \text{rec}} + b_{1}^{\text{sat}} + \lambda_{1} N_{1, \text{rec}}^{\text{sat}} + m_{1} + \varepsilon_{1}^{s} \]

\[ L_{\text{satrec}} = \rho_{\text{rec}}^{\text{sat}} + c \cdot (dt_{\text{rec}} - dt_{\text{sat}}) + T_{\text{rec}}^{\text{sat}} - I_{\text{rec}}^{\text{sat}} + \lambda_{1} \Delta \omega_{\text{rec}}^{\text{sat}} + b_{1, \text{rec}} + b_{1}^{\text{sat}} + \lambda_{1} N_{1, \text{rec}}^{\text{sat}} + m_{1} + \varepsilon_{1}^{s} \]

Differencing between receivers cancels satellite-only-dependent terms

\[ \Delta L_{1}^{\text{sat}} = \Delta \rho^{\text{sat}} - c \Delta dt + \Delta T^{\text{sat}} + \Delta I_{1}^{\text{sat}} + \lambda_{1} \Delta \omega^{\text{sat}} + \Delta b_{1} + \lambda_{1} \Delta N_{1}^{\text{sat}} + m_{\Delta_{1}} + \varepsilon_{\Delta_{1}} \]

\[ \Delta L_{1}^{\text{satR}} = \Delta \rho^{\text{sat}} - c \Delta dt + \Delta T^{\text{satR}} - \Delta I_{1}^{\text{satR}} + \lambda_{1} \Delta \omega^{\text{satR}} + \Delta b_{1} + \lambda_{1} \Delta N_{1}^{\text{satR}} + m_{\Delta_{1}} + \varepsilon_{\Delta_{1}} \]

Differencing between satellites cancels receiver-only dependent terms

\[ \nabla \Delta L_{1} = \nabla \Delta \rho + \nabla \Delta T - \nabla \Delta I_{1} + \lambda_{1} \nabla \Delta \omega + \lambda_{1} \nabla \Delta N_{1} + m_{\Delta \lambda_{1}} + \varepsilon_{\Delta \lambda_{1}} \]

Only residual errors in double differences between sat. and receivers affect relative positioning.
Double-differences between satellites and receivers

\[ \nabla \Delta L_1 = \nabla \Delta \rho + \nabla \Delta T - \nabla \Delta I_1 + \lambda_1 \nabla \Delta \omega + \lambda_1 \nabla \Delta N_1 + mv_{\Delta l} + n_{\Delta l} \]

Satellite and receiver clocks and fractional part of ambiguities cancel.

**Comments:**
- The wind-up term \( \Delta \nabla \omega \) can be neglected, except over long baselines.
- Double-differenced ambiguities are integer numbers of wavelengths.

**Exercise,**
Show that, neglecting the wind-up, the following expressions are met over a continuous carrier phase arch:

\[
\nabla \Delta L_1 - \nabla \Delta \rho \approx \nabla \Delta T - \nabla \Delta I_1 + bias_1 \\
\nabla \Delta L_2 - \nabla \Delta \rho \approx \nabla \Delta T - \gamma \nabla \Delta I_1 + bias_2 \\
\nabla \Delta L_C - \nabla \Delta \rho \approx \nabla \Delta T + bias_C \\
\n\nabla \Delta L_1 - \nabla \Delta L_2 \approx -(1 - \gamma) \nabla \Delta I_1 + bias_1
\]

- Ionosphere-free: Only Troposphere
- Geometry-free: Only Ionosphere

![DGPS DGPS GODS GODN USN3 GODS GODN USN3 GODS GODN USN3 GODS GODN USN3 GODS GODN USN3 GODS GODN USN3 GODS GODN USN3 GODS GODN USN3 GODS GODN USN3 GODS GODN USN3 GODS GODN](image)

Master of Science in GNSS

@ J. Sanz & J.M. Juan
\[ \Delta \nabla (L - \rho) = \Delta \nabla (Tropo + Iono) \]

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@ J. Sanz & J.M. Juan
USN3 23.6 km

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256 m
GNSS Positioning

**Standalone Positioning:** GNSS receiver autonomous positioning using broadcast orbits and clocks (SPS, PPS).

**GNSS Positioning: Space Segment errors**

- **Orbit error**
- **Clock error**

Master of Science in GNSS

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GNSS Positioning: Propagation errors

Ionospheric corrections [Spain, 05/05/2005]

Tropospheric corrections [Spain, 05/05/2005]

Ionosphere

Troposphere

Ionospheric Pierce Points

Permanent Receivers Net.
GNSS Positioning: Propagation errors
Lecture 5
Precise Positioning with carrier phase (PPP)

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5 March 2017
The PPP technique allows centimetre-level accuracy to be achieved for static positioning and decimetre level, or better, for kinematic positioning, after the best part of one hour.

This high accuracy requires the use of code and carrier measurements and an accurate measurement modelling up to centimetre level or better.
Satellite Orbits and Clocks
Broadcast versus precise

- With S/A=on, clocks were degraded several tens of meters.
- Under S/A=off, the broadcast orbits and clocks are accurate at few meters level (see plots at left).
- IGS precise orbits & clocks are accurate at few cm level
IGS Precise orbit and clock products: RMS accuracy, latency and sampling

<table>
<thead>
<tr>
<th>Products</th>
<th>Broadcast (delay)</th>
<th>Broadcast (real time)</th>
<th>Ultra-rapid Predicted (real time)</th>
<th>Observed (3–9 h)</th>
<th>Rapid (17–41 h)</th>
<th>Final (12–18 d)</th>
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<tr>
<td>Orbit GPS</td>
<td>~100 cm (~2 h)</td>
<td>~5 cm (15 min)</td>
<td>~3 cm (15 min)</td>
<td>~2.5 cm (15 min)</td>
<td>~2.5 cm (15 min)</td>
<td>~5 cm (15 min)</td>
</tr>
<tr>
<td>(sampling)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Glonass (sampling)</td>
<td>~5 ns (daily)</td>
<td>~3 ns (15 min)</td>
<td>~150 ps (15 min)</td>
<td>~75 ps (5 min)</td>
<td>~75 ps (30 s)</td>
<td>~ TBD (15 min)</td>
</tr>
</tbody>
</table>

http://igscb.jpl.nasa.gov/components/prods.html
Computation of **satellite coordinates** from precise products.

Precise orbits for GPS satellites can be found on the International GNSS Service (IGS) server [http://igscb.jpl.nasa.gov](http://igscb.jpl.nasa.gov)

Orbits are given by \((x, y, z)\) coordinates with a sampling rate of 15 minutes. The satellite coordinates between epochs can be computed by polynomial interpolation. A 10th-order polynomial is enough for a centimetre level of accuracy with 15 min data.

\[
P_n(x) = \sum_{i=1}^{n} y_i \prod_{j \neq i}^{n} \frac{x - x_j}{x_i - x_j} = y_1 \frac{x - x_2}{x_1 - x_2} \cdots \frac{x - x_n}{x_1 - x_n} + \cdots + y_1 \frac{x - x_1}{x_1 - x_1} \cdots \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdots \frac{x - x_n}{x_i - x_n} + \cdots + y_n \frac{x - x_1}{x_n - x_1} \cdots \frac{x - x_{n-1}}{x_n - x_{n-1}}
\]

---

**IGS orbit and clock products (for PPP):**

Discrepancy between the different centres
Computation of satellite clocks from precise products

Precise clocks for GPS satellites can be found on the International GNSS Service (IGS) server http://igscb.jpl.nasa.gov

IGS is providing precise orbits and clock files with a sampling rate of 15 min, as well as precise clock files with a sample rate of 5 min and 30 sec, in SP3 and CLK formats.

Some centres also provide GPS satellite clocks with a 5 sec sampling rate, like the les obtained from the Crustal Dynamics Data Information System (CDDIS) site.

Stable clocks with a sampling rate of 30 sec or higher can be interpolated with a first-order polynomial to a few centimetres of accuracy. Clocks with a lower sampling rate should not be interpolated, because clocks evolve as random walk processes.
Session 3.2, Ex7a: Precise 300s clock interpolation error

Session 3.2, Ex8: Precise 300s clock interpolation error (SA=on/off)
IGS orbit and clock products (for PPP):
Discrepancy between the different centres

Exercise:
Show that a common error on all satellites does not affect user positioning.

Contents

Precise Point Positioning (PPP)
1. Introduction
2. Orbits and Clocks: Broadcast and Precise
3. Code and carrier measurements and modelling errors
4. Linear observation model for PPP
5. Parameter estimation: Floating Ambiguities
6. Carrier Ambiguity fixing concept: DD and undifferenced
7. Accelerating Filter convergence with accurate ionosphere
For high-accuracy positioning, the carrier phase must be used, besides the code pseudorange.

As commented before, the carrier measurements are very precise, typically at the level of a few millimetres, but contain unknown ambiguities which change every time the receiver locks the signal after a cycle slip.

Nevertheless, such ambiguities can be estimated in the navigation solution, together with the coordinates and other parameters.

- **Code** measurements are unambiguous but noisy (meter level noise).
- **Carrier** measurements are precise (few millimetres of noise) but ambiguous (the unknown biases can reach thousands of km).
- **Carrier phase biases are estimated in the navigation filter** along with the other parameters (coordinates, clock offsets, etc.). If these biases were fixed, measurements accurate to the level of few millimetres would be available for positioning. However, some time is needed to decorrelate such biases from the other parameters in the filter, and the estimated values are not fully unbiased.
Precise Point Positioning: Static

From default configuration of [PPP Template], Select **Static** in the [Filter] panel. **Run gLAB** and plot results.

Receiver positioned as a permanent station (static mode).

Precise Point Positioning: Kinematic

From default configuration of [PPP Template], Select **kinematics** in the [Filter] panel. **Run gLAB** and plot results.

Receiver navigated as a rover in a pure kinematic mode.
In the laboratory session (Tutorial 1) we review the measurements modelling for the Standard Point Positioning (SPP). A brief summary is given next.

After this summary, we will focus on the additional modelling need for Precise Point Positioning.

Remember that the error component most difficult to model is the ionosphere. But, in the PPP technique the ionosphere error is removed (more than 99.9%) using dual-frequency measurements in the ionosphere-free combination (Lc,Pc). This combination also removes the Differential Code Bias (or TGD).
Signal propagation errors on the Atmosphere

Ionosphere error is removed using dual-frequency measurements

Satellite clocks and Total Group Delay (TGD)
The PPP technique allows centimetre-level of accuracy to be achieved for static positioning and decimetre level, or better, for kinematic navigation. This high accuracy requires an accurate modelling “up to the centimetre level or better”, where all previous model terms must be taken into account (except ionosphere and TGD [*]), plus some additional terms given next:

[*] Remember that in the PPP technique the ionosphere error is removed (more than 99.9%) using dual-frequency measurements in the ionosphere-free combination (Lc,Pc). This combination also removes the Differential Code Bias (or TGD).

Additional Modelling for PPP

Satellite Mass Center to Antenna Phase Center

Broadcast orbits are referred to the antenna phase center, but IGS precise orbits are referred to the satellite mass center.
Receiver Antenna Phase center (APC)

GNSS measurements are referred to the APC. This is not necessarily the geometric center of the antenna, and it depends on the signal frequency and the incoming radio signal direction. For geodetic positioning a reference tied to the antenna (ARP) or to monument is used.

Receiver APC:
The antenna used for this experiment, has the APC position vertically shifted regarding ARP. Thence, neglecting this correction, an error on the vertical component occurs, but not in the horizontal one.

Antenna biases and orientation:
- The satellite and receiver antenna phase centres can be found in the IGS ANTEX files, after GPS week 1400 (Nov. 2006)
- The carrier phase wind-up effect due to the satellite’s motion must be taken into account.
Wind-up affects only carrier phase. It is due to the electromagnetic nature of circularly polarized waves of GNSS signals. As the satellite moves along its orbital path, it performs a rotation to keep its solar panels pointing to the Sun direction. This rotation causes a carrier variation, and thence, a range measurement variation.

Wind-Up

Wind-up changes smoothly along continuous carrier phase arcs. In the position domain, wind-up affects both vertical and horizontal components.

High-accuracy GNSS positioning degrades during the GNSS satellites' eclipse seasons.

- Once the satellite goes into shadow, the radiation pressure vanishes. This effect introduces errors in the satellite dynamics due to the difficulty of properly modelling the solar radiation pressure.
- On the other hand, the satellite's solar sensors lose sight of the Sun and the attitude control (trying to keep the panels facing the Sun).

As a consequence, the orbit during shadow and eclipse periods may be considerably degraded and the removal of satellites under such conditions can improve the high-precision positioning results.
Additional Modelling for PPP: **Atmospheric Effects**

- **The ionospheric refraction** and TGDs are **removed** using the ionosphere-free combination of code and carrier measurements (PC, LC).
- **The tropospheric refraction** can be modelled by **Dry and Wet components** (and different mappings are usually used for both components, e.g. mapping of Niell).

![Zenith Tropospheric Delay Estimation](image)

*The troposphere is estimated as a Random Walk process in the Kalman Filter. A process noise of 1cm/sqrt(h) has been taken.*

- **A residual tropospheric delay** is estimated (as wet ZTD delay) in the Kalman filter, together with the coordinates, clock and carrier phase biases.

---

Additional Modelling for PPP: **Earth Deformation**

**Earth deformation effects:**

- **Solid tides** must be modelled by equations
- **Ocean loading and pole tides** are second-order effects and can be neglected for PPP accuracies at the centimetre level

**Solid Tides** concern the movement of Earth’s crust (and thus the variation in the receiver's location coordinates) due to gravitational attractive forces produced by external bodies, mainly the Sun and Moon. Solid tides produce vertical and horizontal displacements that can be expressed by the spherical harmonics expansion \( (m, n) \), characterised by the Love and Shida numbers \( h_{mn} \) and \( l_{mn} \).
Solid Tides
It comprises the Earth’s crust movement (and thence receiver coordinates variations) due to the gravitational attraction forces produced by external bodies, mainly the Sun and the Moon.

These effects do not affect the GNSS signals, but if they were not considered, the station coordinates would oscillate with relation to a mean value. They produce vertical (mainly) and horizontal displacements.

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Precise Point Positioning (PPP)
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3. Code and carrier measurements and modelling errors
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5. Parameter estimation: Floating Ambiguities
6. Carrier Ambiguity fixing concept: DD and undifferenced
7. Accelerating Filter convergence with accurate ionosphere
Linear observation model for PPP

It is based on code and carrier measurements in the ionosphere-free combination (Pc, Lc), which are modelled as follows:

\[ P_{C_{rec}}^{sat} = \rho_{rec}^{sat} + c \cdot (dt_{rec} - dt_{sat}) + Trop_{rec}^{sat} + M_{Pc} + \varepsilon_{Pc} \]
\[ L_{C_{rec}}^{sat} = \rho_{rec}^{sat} + c \cdot (dt_{rec} - dt_{sat}) + Trop_{rec}^{sat} + \lambda_N \omega_{rec}^{sat} + B_{C_{rec}}^{sat} + m_{Lc} + \varepsilon_{Lc} \]

where

\[ P_c = \frac{f_1^2 P_1 - f_2^2 P_2}{f_1^2 - f_2^2} ; \quad L_c = \frac{f_1^2 L_1 - f_2^2 L_2}{f_1^2 - f_2^2} \]
\[ B_C = \lambda_N \left( B_1 + \frac{\lambda_w}{\lambda_2} B_W \right) \]
\[ B_W = B_1 - B_2 \]

Ionosphere is removed

Remark, \( \rho \) is referred to the Antenna Phase Centres (APC) of satellite and receiver antennas in the ionosphere free combination.

Linear model: For each satellite in view

\[ P_{C_{rec}}^{sat} = \rho_{rec}^{sat} + c \cdot (dt_{rec} - dt_{sat}) + Trop + \varepsilon \]

Linearising \( \rho \) around an ‘a priori’ receiver position \( (x_{rec,0}, y_{rec,0}, z_{rec,0}) \)

\[ = \rho_{rec,o}^{sat} + \frac{x_{rec,o} - x_{sat}}{\rho_{rec,o}^{sat}} \Delta x_{rec} + \frac{y_{rec,o} - y_{sat}}{\rho_{rec,o}^{sat}} \Delta y_{rec} + \frac{z_{rec,o} - z_{sat}}{\rho_{rec,o}^{sat}} \Delta z_{rec} + c \cdot (dt_{rec} - dt_{sat}) + Trop \]

where:

\[ \Delta x_{rec} = x_{rec} - x_{rec,o} ; \quad \Delta y_{rec} = y_{rec} - y_{rec,o} ; \quad \Delta z_{rec} = z_{rec} - z_{rec,o} \]

and taking:

\[ Trop_{rec}^{sat} = Trop_{0_{rec}}^{sat} + M_{wet,rec}^{sat} \Delta T_{z,wet} \]

The same for carrier, but adding the ambiguity as an unknown
Linear model: For each satellite in view

\[ P_{C_{\text{sat}}} = \rho_{\text{rec}}^{\text{sat}} + c \cdot (dt_{\text{rec}} - dt_{\text{sat}}) + \text{Trop} + \varepsilon \]

Linearising \( \rho \) around an ‘a priori’ receiver position \((x_{\text{rec},0}, y_{\text{rec},0}, z_{\text{rec},0})\)

\[ \rho_{\text{rec},0}^{\text{sat}} + \frac{x_{\text{rec},0} - x_{\text{sat}}^{\text{sat}}}{\rho_{\text{rec},0}^{\text{sat}}} \Delta x_{\text{rec}}^{\text{sat}} + \frac{y_{\text{rec},0} - y_{\text{sat}}^{\text{sat}}}{\rho_{\text{rec},0}^{\text{sat}}} \Delta y_{\text{rec}}^{\text{sat}} + \frac{z_{\text{rec},0} - z_{\text{sat}}^{\text{sat}}}{\rho_{\text{rec},0}^{\text{sat}}} \Delta z_{\text{rec}}^{\text{sat}} + c (dt_{\text{rec}} - dt_{\text{sat}}) + \text{Trop} \]

where:

\[ \Delta x_{\text{rec}}^{\text{sat}} = x_{\text{rec}}^{\text{sat}} - x_{\text{rec},0} ; \quad \Delta y_{\text{rec}}^{\text{sat}} = y_{\text{rec}}^{\text{sat}} - y_{\text{rec},0} ; \quad \Delta z_{\text{rec}}^{\text{sat}} = z_{\text{rec}}^{\text{sat}} - z_{\text{rec},0} \]

and taking:

\[ Trop_{\text{rec}}^{\text{sat}} = Trop_{0\text{rec}}^{\text{sat}} + M_{\text{wet,rec}}^{\text{wet}} \Delta T_{rZ,wet}^{\text{sat}} \]

**Prefit-residuals (Prefit)**

Following the same procedure as with the code based positioning (SPP), the linear observation model \( y = G x \) for the code and carrier measurements can be written as follows:

\[ y = G x \]

\[ \begin{bmatrix} P_{\text{Prefit(C)}}^{0} \\ P_{\text{Prefit(L)}}^{0} \\ \vdots \\ P_{\text{Prefit(C)}}^{n} \\ P_{\text{Prefit(L)}}^{n} \end{bmatrix} = \begin{bmatrix} x_{0,\text{rec}} - x_{1}^{1} & y_{0,\text{rec}} - y_{1}^{1} & z_{0,\text{rec}} - z_{1}^{1} & 1 & M_{\text{wet}}^{n} & 0 & \cdots & 0 \\ x_{0,\text{rec}} - x_{1}^{1} & y_{0,\text{rec}} - y_{1}^{1} & z_{0,\text{rec}} - z_{1}^{1} & 1 & M_{\text{wet}}^{n} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{0,\text{rec}} - x_{n}^{n} & y_{0,\text{rec}} - y_{n}^{n} & z_{0,\text{rec}} - z_{n}^{n} & 1 & M_{\text{wet}}^{n} & 0 & \cdots & 0 \\ x_{0,\text{rec}} - x_{n}^{n} & y_{0,\text{rec}} - y_{n}^{n} & z_{0,\text{rec}} - z_{n}^{n} & 1 & M_{\text{wet}}^{n} & 0 & \cdots & 1 \end{bmatrix} \]

\[ \begin{bmatrix} \Delta x_{\text{rec}}^{\text{sat}} \\ \Delta y_{\text{rec}}^{\text{sat}} \\ \Delta z_{\text{rec}}^{\text{sat}} \\ \Delta T_{rZ,wet}^{\text{sat}} \\ B_{C}^{k} \\ \vdots \\ B_{C}^{\alpha} \end{bmatrix} \]

**Prefit (P)**

\[ P_{\text{Prefit(C)}}^{k} = P_{C}^{k} - \rho_{0}^{k} + cdt^{k} - Trop_{0}^{k} \]

**Prefit (L)**

\[ P_{\text{Prefit(L)}}^{k} = L_{C}^{k} - \rho_{0}^{k} + cdt^{k} - Trop_{0}^{k} - \lambda_{N} \omega^{k} \]

Carrier ambiguities
Equivalent notation:

Using \( \rho = \rho_0 - \hat{\rho}^T \cdot \Delta r \), where \( \hat{\rho} = \frac{\rho_0}{\rho_0} \)

The previous system, can be written as:

\[
\begin{bmatrix}
\text{Prefit}(Pc)^n \\
\text{Prefit}(Lc)^n
\end{bmatrix}
= 
\begin{bmatrix}
\hat{\rho}^T_{0 \text{rec}} \\
\hat{\rho}^T_{0 \text{rec}}
\end{bmatrix} 
\begin{bmatrix}
M_{\text{wet}}^1 \\
M_{\text{wet}}^1
\end{bmatrix}
\begin{bmatrix}
0 & \cdots & 0 \\
0 & \cdots & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Delta r_{\text{rec}} \\
cdt_{\text{rec}} \\
\Delta T_{r,wet}
\end{bmatrix}
\]

\[
\begin{bmatrix}
B^1 \\
\vdots \\
B^n
\end{bmatrix}
\]

Carrier ambiguities

\[
\begin{aligned}
\text{Prefit}(Pc)^k &= P_c^k - \rho_0^k + cdt^k - Trop_0^k \\
\text{Prefit}(Lc)^k &= L_c^k - \rho_0^k + cdt^k - Trop_0^k - \lambda \omega^k
\end{aligned}
\]

---

**Contents**

Precise Point Positioning (PPP)

1. Introduction
2. Orbits and Clocks: Broadcast and Precise
3. Code and carrier measurements and modelling errors
4. Linear observation model for PPP
5. Parameter estimation: Floating Ambiguities
6. Carrier Ambiguity fixing concept: DD and undifferenced
7. Accelerating Filter convergence with accurate ionosphere
Parameter estimation PPP: Floating Ambiguities

The linear observation model $y = G \mathbf{x}$ can be solved using the Kalman filter. The next stochastic model can be used

- **Carrier phase biases** ($B_C$) are taken as `constant´ along continuous phase arcs, and as `white noise´ when a cycle slip happens ($\sigma = 10^4 m$ can be taken, for instance) → **FLOATED AMB.**
- **Wet tropospheric delay** ($Tr_{z,wet}$) is taken as a random walk process (with $d\sigma^2/dt = 1 \text{ cm}^2/\text{h}$, for instance).
- **Receiver clock** ($c_{dt}$) is taken as a white-noise process (with $\sigma = 3.10^5 m$, i.e. $1 \text{ ms}$ for instance).
- **Receiver coordinates** ($\Delta x, \Delta y, \Delta z$)
  - For **static positioning** the coordinates are taken as constants.
  - For **kinematic positioning** the coordinates are taken as white noise or a random walk process.

---

**Comment: Floating Ambiguities**

This solution procedure is called **floating ambiguities**. `Floating´ in the sense that the ambiguities are estimated by the filter `as real numbers´.

The bias estimations $\hat{B}_C$ will converge to a solution after a transition time that depends on the observation geometry, model quality and data noise.

In general, one must expect errors at the decimetre level, or better, in pure kinematic positioning (after the best part of one hour) and at the centimetre level in static PPP over 24h data.

See exercises in the laboratory session (Tutorial 1).
In the previous formulation, the ambiguities have been estimated as real numbers in the Kalman filter, without exploiting its integer nature. That is, the ambiguities in $L_1$, $L_2$ or $L_W$ signals are an integer number ($N$) of its associated wavelength ($\lambda$) plus a fractional part associated to the satellite and to the receiver.

$$B_{1,\text{sat}}^{\text{sat}} = \lambda_1 N_{1,\text{rec}}^{\text{sat}} + b_{1,\text{rec}} + b_{1}^{\text{sat}}$$
$$B_{2,\text{rec}}^{\text{sat}} = \lambda_2 N_{2,\text{rec}}^{\text{sat}} + b_{2,\text{rec}} + b_{2}^{\text{sat}}$$
$$N_W = N_1 - N_2$$

$$B_C = \lambda_N \left( N_1 + \frac{\lambda_W}{\lambda_2} N_W \right) + b_{C,\text{rec}} + b_C^{\text{sat}}$$

$B_C$ is not an integer number of $\lambda_N$ but can be related with the integers $N_1$, $N_W$.

The Ambiguity Fixing techniques take benefit of this INTEGER NATURE of $N_1$, $N_2$ ambiguities to properly fix them, reducing convergence time, and thence, achieving high accuracy quickly.
Two different approaches can be considered:

- **Double differenced Ambiguity Fixing (e.g. RTK):**
  This is the classical approach which relies on the fact that the fractional part of carrier ambiguities cancels when forming the double differences between receivers and satellites:

  ➔ That is, given:
  \[
  B_{rec}^{sat} = \lambda N_{rec}^{sat} + b_{rec} + b_{sat}^{sat}
  \]

  The double differences, regarding a reference receiver and satellite, yield:
  \[
  \Delta \nabla B_{rec}^{sat} = B_{rec} - B_{rec,R}^{sat} - (B_{rec, R}^{sat} - B_{rec, R}^{sat}) = \lambda \Delta \nabla N_{rec}^{sat}
  \]

  where the satellite and receiver ambiguity terms \((b_{rec}, b_{sat})\) cancel out.

- **Absolute (or undifferenced) ambiguity fixing (e.g. PPP AR):**

---

**Example of Ambiguity Fixing by rounding**

![Graph showing DDNw ambiguity for PRN03](image)

- **DDNw ambiguity:** PRN03
- **Cycles LW:**
  - 3
  - 4
  - 5
  - 6
  - 7

- **Time (s):**
  - 18000
  - 18500
  - 19000
  - 19500
  - 20000
Two different approaches can be considered:

• **Double differenced Ambiguity Fixing (e.g. RTK):**
  This is the classical approach which relies in the fact that the fractional part of carrier ambiguities cancels when forming the double differences between receivers and satellites:

  ➔ That is, given:

  \[
  B_{\text{rec}}^{\text{sat}} = \lambda N_{\text{rec}}^{\text{sat}} + b_{\text{rec}} + B_{\text{sat}}^{\text{rec}}
  \]

  The double differences, regarding a reference receiver and satellite, yield:

  \[
  \Delta B_{\text{rec}}^{\text{sat}} = B_{\text{rec}}^{\text{sat}} - B_{\text{rec},R}^{\text{sat}} - \left( B_{\text{sat},R}^{\text{rec}} - B_{\text{sat},R}^{\text{rec}} \right) = \lambda \Delta N_{\text{rec}}^{\text{sat}}
  \]

  **Absolute (or undifferenced) ambiguity fixing (e.g. PPP AR):**
  This is a new approach, where the fractional part of the ambiguities are estimated from a global network of permanent stations and provided to the users. Thus, the user can remove this fractional part and fix the remaining ambiguity as an integer number.

  \[
  B_{\text{rec}}^{\text{sat}} = \lambda N_{\text{rec}}^{\text{sat}} + b_{\text{rec}} + B_{\text{sat}}^{\text{rec}}
  \]
Absolute (or undifferenced) ambiguity fixing

Nevertheless, these carrier hardware biases (or fractional part of carrier ambiguities) can be estimated together with the orbits, clocks, ionosphere and DCBs from measurements of a worldwide reference stations network.

These carrier hardware biases are slow varying parameters and can be broadcast to the user, together with the other differential corrections (orbits, clocks, ionosphere...) from a reference station network. Then, the user can remove them and estimate the remaining ambiguities as integer numbers.

This can allow world-wide PPP users to perform ambiguity fixing, without baseline length limitation, improving accuracy and reducing convergence time.

Note: These carrier hardware biases are canceled in RTK when forming Double-Differences of measurements between pairs of satellites and receivers.

The main weakness of PPP is the large convergence time, which depend on satellites geometry, quality of data (code noise, cycle-slips...). There is a sudden improvement when fixing carrier ambiguities, but still several tens of minutes are needed to achieve centimeter level of accuracy.

Fast-PPP: Accelerating convergence with IONO

• During the early 2000s gAGE/UPC developed a two layers grid model, and afterwards assessed its suitability to help the PPP convergence.

24 November 2009 (low solar activity)

These figures are from (Juan et al, 2012)

More recently, the accurate ionospheric model has been extended to worldwide (Rovira, et al 2014)

MLVL (252km), EUSK (170km) and EIJS (95km) (km from the nearest reference receiver)
**How the ionosphere helps the filter convergence in Fast-PPP?**

1) There is a constraint between the wide-lane ambiguity \(B_w\), the  iono-free ambiguity \(B_d\) and the **ionospheric refraction** \(STEC\) and \(DCB\).

\[
B_C = B_w - 1.98 (L_1 - L_2 - STEC - DCB)
\]

2) The **wide-lane** ambiguity is estimated/fixed quickly (~5 minutes with 2-freq and in single epoch with 3-freq. signals).

3) The **ionosphere** \(STEC\) is the bridge (through the mentioned constraint) to transfer the quick accuracy achieved in the **wide-lane ambiguity** \(B_w\) computed value to the **iono-free ambiguity** \(B_d\) estimation, accelerating in this way the filter convergence.

Thence, in Fast-PPP the convergence time is strongly reduced thanks to the quick wide-lane ambiguity fixing and the accurate ionospheric corrections. It allows to achieve High Accuracy quickly (~5 minutes with 2-freq & single epoch with 3-freq).

The ionosphere will help provided that its quality (noise/error) is better than the code noise.

This is a conceptual approach!
References


Thank you

Backup slides
Fast-PPP Ionospheric Model

1.- Derived from the geometry-free combination of carrier-phases:

\[ LI = L1 - L2 \]

2.- Ambiguities are fixed (thanks to the CPF accurate geodetic modelling):

Unambiguous

\[ LI_i^j - BI_i^j = STEC_i^j + DCB_i - DCB_j \]

3.- The Slant Total Electron Content (STEC) is estimated with Vertical STEC (VTEC) on each “k” Ionospheric Grid Point (IGP) at the two-layers:

\[ STEC_i^j = \sum \alpha_k \cdot VTEC_k \]

2-Layer Ionospheric Maps during Storm of DoY 058-2014

- 1st Layer: Ionosphere 270 km
- 2nd Layer: Plasmasphere 1600 km

Orbit errors (4cm) + sat Clocks (6cm)
Iono Accuracy <= 1 TECU (16 cm in L1)
Agreement Electron Contents of dual-layers between GIM derived from “earth” data and “space-borne” data

**Independent data**


1 day of gAGE dual-layer GIM.

TEC ratio:

\[
\text{IononeTEC} = \frac{\text{IononeTEC + PlasmaTEC}}{2}
\]

Neglecting this effect, introduce a miss-modelling degrading navigation.

---


---

**• Code** measurements are unambiguous but noisy (meter level noise).

**• Carrier** measurements are precise (few millimetres of noise) but ambiguous (the unknown ambiguities can reach thousands of km).

**• Carrier phase ambiguities are estimated in the navigation filter** along with the other parameters (coordinates, clock offsets, etc.). If these ambiguities were fixed, measurements accurate to the level of few millimetres would be available for positioning. However, some time is needed to decorrelate such ambiguities from the other parameters in the filter, and the estimated values are not fully unbiased.
Convergence: DoYs 169-200 of 2014
6 Positioning modes per rover, resets every 2h:
   a) Reference Iono-Free sol
   b) Iono Sol: IONEX (IGS-GIM)
   c) Iono Sol. Fast-PPP IONO:

Rover EIJS (5°E, 50°N): Mid Latitude, 76 km
Results from (Rovira et al., 2014)

Rover LKHU (95°W, 29°S): Low latitude, 455 km

Benefits of applying external ionospheric corrections are expected, provided its quality is superior to that of code noise.
Lecture 6
Differential positioning with Code pseudoranges

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Authorship statement

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5 March 2017
1. Linear model for DGNSS: Single Differences
   1.1. Linear model
   1.2. Geographic decorrelation of ephemeris errors
   1.3. Error mitigation and ‘short’ baseline concept
   1.4. Differential code based positioning

2. Augmentation Systems
   2.1. Introduction
   2.2. Ground-Based Augmentation system (GBAS)
   2.3. Satellite based Augmentation system (SBAS)

Error mitigation: DGNSS residual error

Errors are similar for users separated tens, even hundred of kilometres, and these errors vary ‘slowly’ with time. That is, the errors are correlated on space and time.

The spatial decorrelation depends on the error component (e.g. Clocks not decorrelate, ionosphere ~100km...). Thence, long baselines need a reference stations network.
Linear model for Differential Positioning

Code and carrier measurements

\[
P^i_j = \rho^i_j + c(\delta t_i - \delta t^j_i) + T^i_j + I^i_j + K_i + K^j_i + M^i_j + \nu_{p_i}^j
\]
\[
L^i_j = \rho^i_j + c(\delta t_i - \delta t^j_i) + T^i_j - I^i_j + \lambda \omega^j_i + \lambda N^i_j + b^i + b^j_i + m^i_j + \nu_{l_i}^j
\]

When approximate values of both receiver and satellite APC positions are taken, a linearization around them yields:

\[
\rho^i_j = \rho^i_{0i} - \hat{\rho}^i_{0i} \cdot \Delta r_i + \hat{\rho}^j_{0i} \cdot \Delta r^j
\]

\[
\hat{\rho}^j_{0i} = \frac{r^j_i - r^j_{0i}}{\| r^j_i - r^j_{0i} \|}
\]

\[\Delta r_i : \text{Receiver coordinates error}\]
\[\Delta r^j_i : \text{Satellite coordinates error}\]

Linear model for Differential Positioning

Code and carrier measurements

\[
P^i_j = \rho^i_j + c(\delta t_i - \delta t^j_i) + T^i_j + I^i_j + K_i + K^j_i + M^i_j + \nu_{p_i}^j
\]
\[
L^i_j = \rho^i_j + c(\delta t_i - \delta t^j_i) + T^i_j - I^i_j + \lambda \omega^j_i + \lambda N^i_j + b^i + b^j_i + m^i_j + \nu_{l_i}^j
\]

Single difference

\[\Delta (\bullet)_n = (\bullet)_u - (\bullet)_r\]

\[
P^u_j = \rho^u_j + c(\delta t_u - \delta t^j_u) + T^u_j + I^u_j + K_u + K^j_u + \nu_{p_u}^j
\]
\[
P^r_j = \rho^r_j + c(\delta t_r - \delta t^j_r) + T^r_j + I^r_j + K_r + K^j_r + \nu_{p_r}^j
\]

\[
P^i_{ru} = \rho^i_{ru} + c(\delta t_{ru} + T^i_{ru} + I^i_{ru} + K_{ru} + \nu_{p_{ru}}^j
\]

The same for the carrier:

\[
L^i_{ru} = \rho^i_{ru} + c(\delta t_{ru} + T^i_{ru} - I^i_{ru} + \lambda \omega^j_{ru} + \lambda N^i_{ru} + b_{ru} + \nu_{l_{ru}}^j
\]
Linear model for Differential Positioning

**Code and carrier measurements**

\[
P^j_i = \rho^j_i + c(\delta t^j_i - \delta t^i_i) + T^j_i + I^j_i + K^j_i + M^j_i + v^j_{pr}
\]

\[
L^j_i = \rho^j_i + c(\delta t^j_i - \delta t^i_i) + T^j_i - I^j_i + \lambda \omega^j_i + \lambda N^j_i + b_i + b^j_i + m^j_i + v^j_{z_i}
\]

**Single difference** \( \equiv \Delta(\cdot)^j_i = (\cdot)^j_i - (\cdot)^j_r \)

\[
P^j_{ru} = \rho^j_{ru} + c \delta t^j_{ru} + T^j_{ru} + I^j_{ru} + K_{ru} + v^j_{pr}
\]

\[
L^j_{ru} = \rho^j_{ru} + c \delta t^j_{ru} + T^j_{ru} - I^j_{ru} + \lambda \omega^j_{ru} + \lambda N^j_{ru} + b_{ru} + v^j_{z_{ru}}
\]

**Single difference cancels:**
- Satellite clock \( (\delta t^j_i) \)
- Satellite code instrumental delays \( (K^j_i) \)
- Satellite carrier instrumental delays \( (b^j_i) \)

**Single differences mitigate/remove errors due**
- Satellite Ephemeris \( (\Delta r^i) \)
- Ionosphere \( (I^j_i) \)
- Troposphere \( (T^j_i) \)
- Wind-up \( (\omega^j_i) \)

The residual errors will depend upon the baseline length.

---

**Single-Difference of measurements**

(corrected by geometric range!!)

\[
\Delta(L^j - \rho) = L^j_{ru} - \rho^j_{ru}
\]

\[
\Delta(P^j - \rho) = P^j_{ru} - \rho^j_{ru} = c \delta t^j_{ru} + T^j_{ru} - I^j_{ru} + \lambda \omega^j_{ru} + \lambda N^j_{ru} + b_{ru} + v^j_{z_{ru}}
\]

**Dif. Receiver clock:**
Main variations *Common for all satellites*

**Dif. Tropo. and Iono.:**
Small variations

**Dif. Instrumental delays and carrier ambiguities:**
constant

Dif. Wind-up: Very small
Single-Difference of measurements (corrected by geometric range!!)

\[ \Delta(L_1 - \rho) \equiv L_{ru}^{\text{sat}} - \rho_{ru}^{\text{sat}} \]

\[ \Delta(P_1 - \rho) \equiv P_{ru}^{\text{sat}} - \rho_{ru}^{\text{sat}} \]

Dif. Receiver clock:
Main variations Common for all satellites

Dif. Tropo. and Iono.:
Small variations

Dif. Instrumental delays and carrier ambiguities:
constant

Dif. Wind-up: Very small
Linear model for Differential Positioning

Code and carrier measurements

\[
P_i^j = \rho_i^j + c(\delta t_i - \delta t^j) + T_i^j + I_i^j + K_i + K^j + \nu_r^j
\]

\[
L_i^j = \rho_i^j + c(\delta t_i - \delta t^j) + T_i^j - I_i^j + \lambda \omega_i^j + \lambda N_i^j + b_i + b^j + \nu_{r_1}^j
\]

where:

\[
\rho_i^j = \rho_{oi}^j - \hat{\rho}_{0i}^j \cdot \Delta r_i + \hat{\rho}_{0i}^j \cdot \Delta r^j
\]

**Single difference**

\[
(\bullet)^j_{ru} \equiv \Delta(\bullet)^j_{ru} = (\bullet)^j_{u} - (\bullet)^j_{r}
\]

\[
P^j_{ru} = \rho^j_{ru} + c(\delta t_{ru} + T^j_{ru} + I^j_{ru} + K_{ru} + \varepsilon^j_{ru})
\]

\[
L^j_{ru} = \rho^j_{ru} + c(\delta t_{ru} + T^j_{ru} - I^j_{ru} + \lambda \omega_{ru}^j + \lambda N_{ru}^j + b_{ru} + \nu_{ru}^j)
\]

where:

\[
\rho^j_{ru} = \rho^j_{0ru} - \hat{\rho}_{0ru}^j \cdot \Delta r_{ru} - \hat{\rho}_{0ru}^j \cdot \varepsilon_{site} + \hat{\rho}_{0ru}^j \cdot \varepsilon_{eph}
\]

being:

\[
\rho^j_{0ru} \equiv \rho^j_{0u} - \rho^j_{0r}, \quad \Delta r_{ru} \equiv \Delta r_{u} - \varepsilon_{site}
\]

Finally:

\[
\mathbf{r}_u = \mathbf{r}_{0u} + \Delta \mathbf{r}_{ru} + \varepsilon_{site}
\]

Linear model for Differential Positioning

and where the \(\Delta \mathbf{r}_{ru}\) estimate will be affected in turn by the ephemeris and sitting site errors as:

\[
\rho^j_{ru} = \rho^j_{0ru} - \hat{\rho}_{0ru}^j \cdot \Delta \mathbf{r}_{ru} - \hat{\rho}_{0ru}^j \cdot \varepsilon_{site} + \hat{\rho}_{0ru}^j \cdot \varepsilon_{eph}
\]

Range error due to reference station coordinates uncertainty

Range error due to Sat. coordinates uncertainty

Thence, taking into account the relationships:

\[
\hat{\rho}_{0ru}^j \cdot \varepsilon_{site} \leq \frac{b}{\rho^j_u} \| \varepsilon_{site} \| ; \quad \hat{\rho}_{0ru}^j \cdot \varepsilon_{eph} \leq \frac{b}{\rho^j_u} \| \varepsilon_{eph} \|
\]

and being \(\rho^j_u \approx 20000\text{ km}\) it follows that for a baseline \(b = 20\text{ km}\)

\[
\frac{b}{\rho^j_u} \approx \frac{1}{1000} \Rightarrow \text{The effect of 5 metres error in orbits or in site coordinates is less than 5 mm in range for the estimation of } \Delta \mathbf{r}_{ru} .
\]

But, the user position estimate will be shifted by the error in the site coordinates \(\mathbf{r}_u = \mathbf{r}_{0u} + \Delta \mathbf{r}_{ru} + \varepsilon_{site}\)
Exercise:

Let be: 
\[
\rho_{ru}^j = \rho_u^j - \rho_r^j
\]
where
\[
\rho_i^j = \rho_{oi}^j - \hat{\rho}_{oi}^j \cdot \Delta r_i + \hat{\rho}_{oi}^j \cdot \Delta r^j
\]
(from Taylor expansion)

Show that the Single Differences are given by:
\[
\rho_{ru}^j = \rho_{0ru}^j - \hat{\rho}_{0u}^j \cdot \Delta r_{ru} - \hat{\rho}_{0ru}^j \cdot \Delta r_r + \hat{\rho}_{0ru}^j \cdot \Delta r^j
\]

being:
\[
\rho_{0ru}^j \equiv \rho_{0u}^j - \rho_{0r}^j; \quad \Delta r_{ru} \equiv \Delta r_u - \Delta r_r
\]

Hint: Linear model for Differential Positioning

Code and carrier measurements
\[
P_i^j = \rho_i^j + c(\delta t_i - \delta t^j) + T_i^j + I_i^j + K_i + K_i^j + v^j
\]
\[
L_i^j = \rho_i^j + c(\delta t_i - \delta t^j) + T_i^j - I_i^j + \lambda \omega_i^j + \lambda N_i^j + b_l + b^j + v_{\ell}^j
\]

where:
\[
\rho_i^j = \rho_{oi}^j - \hat{\rho}_{oi}^j \cdot \Delta r_i + \hat{\rho}_{oi}^j \cdot \Delta r^j
\]
Linear model for Differential Positioning

\[ \rho_{ru}^j = \rho_{0ru}^j - \hat{\rho}_{0u}^j \cdot \Delta \mathbf{r}_{ru} - \hat{\rho}_{0ru}^j \cdot \Delta \mathbf{r}_r + \hat{\rho}_{0ru}^j \cdot \Delta \mathbf{r}^j \]

with \( \Delta \mathbf{r}_{ru} \equiv \Delta \mathbf{r}_u - \Delta \mathbf{r}_r \)

Let’s assume that:

- The satellite coordinates are known with an uncertainty \( \Delta \mathbf{r}^j \equiv \mathbf{e}_{eph}^j \)
- The reference station coordinates are known with an uncertainty \( \Delta \mathbf{r}_r \equiv \mathbf{e}_{site} \) (i.e. \( \mathbf{r}_r = \mathbf{r}_{0r} + \mathbf{e}_{site} \))

Thence, the user position can be computed from the estimate, with an error \( \mathbf{e}_{site} \), as:

\[ \mathbf{r}_u = \mathbf{r}_{0u} + \Delta \mathbf{r}_{ru} + \mathbf{e}_{site} \]

and where the \( \Delta \mathbf{r}_{ru} \) estimate will be affected in turn by the ephemeris and sitting site errors as:

\[ \rho_{ru}^j = \rho_{0ru}^j - \hat{\rho}_{0u}^j \cdot \Delta \mathbf{r}_{ru} - \hat{\rho}_{0ru}^j \cdot \mathbf{e}_{site} + \hat{\rho}_{0ru}^j \cdot \mathbf{e}_{eph}^j \]

Range error due to reference station coordinates uncertainty

Range error due to Sat. coordinates uncertainty

Thence, taking into account the relationships:

\[ \hat{\rho}_{0ru}^j \cdot \mathbf{e}_{site} \leq \frac{b}{\rho_{u}^j} ||\mathbf{e}_{site}|| ; \quad \hat{\rho}_{0ru}^j \cdot \mathbf{e}_{eph}^j \leq \frac{b}{\rho_{u}^j} ||\mathbf{e}_{eph}^j|| \]

and being \( \rho_{u}^j \approx 20000 \text{km} \) it follows that for a baseline \( b = 20 \text{ km} \)

\[ \frac{b}{\rho_{u}^j} \approx \frac{1}{1000} \Rightarrow \]

The effect of 5 metres error in orbits or in site coordinates is less than 5 mm in range for the estimation of \( \Delta \mathbf{r}_{ru} \).

But, the user position estimate will be shifted by the error in the site coordinates \( \mathbf{r}_u = \mathbf{r}_{0u} + \Delta \mathbf{r}_{ru} + \mathbf{e}_{site} \)
Exercise:
Demonstrate the following relationship:

\[ \hat{\rho}_{0ru} \cdot \varepsilon_{site} \leq \frac{b}{\rho} \| \varepsilon_{site} \| \]

**Hint:**

\[ \hat{\rho}_{0ru} \cdot \varepsilon_{site} = \left( \hat{\rho}_{0u} - \hat{\rho}_{0r} \right) \cdot \varepsilon_{site} \]

\[ \approx \left( \frac{\rho_{0u} - \rho_{0r}}{\rho} \right) \cdot \varepsilon_{site} \leq \frac{b}{\rho} \| \varepsilon_{site} \| \]

**Note:** The following approaches have been taken:

\[ \rho_{0r} \approx \rho_{0u} \approx \hat{\rho} \Rightarrow \]

\[ \begin{align*}
\hat{\rho}_{0u} &= \frac{\rho_{0u}}{\rho} \\
\hat{\rho}_{0r} &= \frac{\rho_{0r}}{\rho} \\
\hat{\rho}_{0r} &= \frac{\rho_{0r}}{\rho} \\
\end{align*} \]

\[ b = \| r_n \| \approx \| r_{0ru} \| \quad u \cdot v \leq \| u \| \| v \| \]

---

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Geographic decorrelation of ephemeris errors

True position

Satellite location error $\varepsilon$

Differential range error due to satellite obit error

$$\delta \rho = \frac{\varepsilon \cdot \hat{\rho}_{\text{user}} - \varepsilon \cdot \hat{\rho}_{\text{ref}}}{\rho_{\text{user}} \rho_{\text{ref}}}$$

A conservative bound:

$$\delta \rho < \frac{b}{\rho}$$

with a baseline $b = 20km$

$$\delta \rho < \frac{20}{20000} = \frac{1}{1000}$$

Reference Station

User

Baseline: $b$

Satellite location error $\varepsilon$
**Orbit Errors over the hyperboloid will not produce differential range errors.**

\[
\tilde{\rho}_1 = \rho_1 + \Delta \rho_{\text{eph}} \\
\tilde{\rho}_2 = \rho_2 + \Delta \rho_{\text{eph}} \\
\Rightarrow \tilde{\rho}_1 - \tilde{\rho}_2 = \rho_1 - \rho_2 = \text{constant} \Rightarrow \delta \rho = 0
\]
Orbit Errors over the hyperboloid will not produce differential range errors.

\[ \tilde{\rho}_1 = \rho_1 + \Delta \rho_{\text{eph}} \]
\[ \tilde{\rho}_2 = \rho_2 + \Delta \rho_{\text{eph}} \]

\[ \Rightarrow \tilde{\rho}_1 - \tilde{\rho}_2 = \rho_1 - \rho_2 = \text{constant} \Rightarrow \delta \rho = 0 \]

Errors over the hyperboloid (i.e. \( \rho_B - \rho_A = ctt \)) will not produce differential range errors.

The highest error is given by the vector \( \hat{u} \), orthogonal to the hyperboloid and over the plain containing the baseline vector \( \hat{b} \) and the LoS vector \( \hat{\rho} \).

Note:
Being the baseline \( \hat{b} \) much smaller than the distance to the satellite, we can assume that the LoS vectors from A and B receives are essentially identical to \( \hat{\rho} \). That is, \( \rho_B \approx \rho_A \approx \rho \).

\[ \hat{u} = \hat{\rho} \times (\hat{b} \times \hat{\rho}) = \hat{b} (\hat{\rho}^T \cdot \hat{b}) \cdot (\hat{\rho}^T \cdot \hat{b}) \]
\[ = \hat{b} - (\hat{\rho} \cdot \hat{\rho}^T) \hat{b} = (1 - \hat{\rho} \cdot \hat{\rho}^T) \hat{b} \]

Note: \( \hat{u} = \sin \phi \hat{\rho} \)

Differential range error \( \delta \rho \) produced by an orbit error \( \hat{e}_1 \) parallel to vector \( \hat{u} \)

Let \( \delta \epsilon \equiv \hat{e}_1 \)

\[ \delta \rho = \delta (\rho_B - \rho_A) = 2 \delta a = 2 \frac{\partial a}{\partial \epsilon} \delta \epsilon = 2 \frac{\partial a}{\partial \epsilon} \frac{\partial \phi}{\partial \epsilon} = -b \sin \phi \frac{\partial \phi}{\partial \epsilon} \]
\[ \approx -b \sin \phi \frac{1}{\rho} \frac{\partial \epsilon}{\partial \epsilon} \]

Note: \( \hat{\epsilon} \perp \hat{b} \Rightarrow \delta \epsilon = \rho \delta \phi \)

Thence:

\[ \delta \rho = -b \sin \phi \epsilon^T \cdot \hat{u} = -\epsilon^T (\sin \phi \hat{u}) \frac{b}{\rho} \]
\[ = -\epsilon^T (1 - \hat{\rho} \cdot \hat{\rho}^T) \frac{b}{\rho} \]

Where: \( b = \hat{b} \hat{b} \)

is the baseline vector
ORB: Orbit Test
Broadcast orbits
Along-track Error (PRN17)

PRN17:
Doy=077, Transm. time: 64818 sec

17 10 3 18 20 0 0.0 1.379540190101E-04 2.84217893040E-12 0.0000000000000E+00
7.8000000000000E+01-5.8593750000000E+01 4.50693447820E-09-2.983492318682E+00
-9.25797653169E-05 5.277505260892E-03 8.186325430870E-06 5.15357815361E+03
4.1760000000000E+03-5.401670932770E-08-4.048348681654E-01-7.636845111847E-08
9.69368515702E-01 2.2153125000000E+02-2.547856683066E+00-7.964974630307E-09
-3.771585673111E-10 1.0000000000000E+00 1.5750000000000E+03 0.0000000000000E+00
2.0000000000000E+03 0.0000000000000E+00-1.624454832787E+00 7.8000000000000E+00
4.1041800000000E+05 4.0000000000000E+01

diff EPH.dat.org EPHcuc_x0.dat
< -2.579763531685E-04 5.277505260892E-03 8.186325430870E-06 5.15357815361E+03
> -9.25797653169E-05 5.277505260892E-03 8.186325430870E-06 5.15357815361E+03

Orbit error $\vec{e}$
Range error $\rho$
Differential range error
Baseline: $b=31.3\text{km}$
Exercise:

Justify that clock errors completely cancel in differential positioning.

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Error mitigation and short baseline concept

If the distance between the user and the reference station is "short enough", so that the residual error ionospheric, tropospheric and ephemeris are small compared to the typical errors due to receiver noise and multipath, it can be assumed:

\[ T_{ru}^j = I_{ru}^j = 0; \quad \hat{\rho}_{0ru}^j \cdot \epsilon_{eph}^j = 0 \]

Note that the previous definition of “shortness” is quite fuzzy

Working with smoothed code, a residual error of about 0.5 metres could be tolerable, but for carrier based positioning it should be less than 1 cm to allow the carrier ambiguity fixing.

- The differential ephemeris error is at the level of few centimetres for baselines up to 100 km (i.e. 5 cm assuming a large bound of \( \epsilon_{eph}^j \leq 10 \text{ m} \)).

- The typical spatial gradient of the ionosphere (STEC) is 1-2 mm/km (i.e. 0.1-0.2 m in 100km), but it can be more than one order of magnitude higher when the ionosphere is active.

Note that the previous definition of “shortness” is quite fuzzy

- The correlation radio of the troposphere is lower than for the ionosphere. At 10km of separation the residual error can be up to 0.1-0.2 m. Nevertheless, 90% of the tropospheric delay can be modelled and the remaining 10% can be estimated together with the coordinates (for high precision applications). For distances beyond a ten of kilometres or significant altitude difference it would be preferable to correct for the tropospheric delay at the reference station and user receiver.

Carrier-smoothed code:
Pseudorange code measurement errors due to receiver noise and multipath can be reduced smoothing the code with carrier measurements. Smoothed codes of 0.5m (RMS) can be obtained with 100 seconds smoothing. On the other hand, the ionospheric error is substantially eliminated in differential mode and the filter can be allowed for larger time smoothing windows.
Differential code based positioning

If the reference station coordinates are known at the centimetre level and the distance between reference station and user are "not too large", we can assume

\[
\begin{align*}
T_{ru}^j & \approx 0 ; I_{ru}^j \approx 0 \\
\hat{p}_{0ru}^j \cdot \epsilon_{eph}^j & \approx 0 \\
\epsilon_{site}^0 & \approx 0 \Rightarrow \Delta r_{ru} \approx \Delta r_u
\end{align*}
\]

Thence,

\[
\begin{align*}
P_{ru}^j &= \rho_{ru}^j + c \delta t_{ru} + T_{ru}^j + K_{ru} + \nu_{\rho ru}^j \\
\rho_{ru}^j &= \rho_{0ru}^j - \hat{\rho}_{0ru}^j \cdot \Delta r_{ru} - \hat{\rho}_{0ru}^j \cdot \epsilon_{site} + \hat{\rho}_{0ru}^j \cdot \epsilon_{eph} \\
\end{align*}
\]

\[
\begin{align*}
P_{ru}^j &= \rho_{ru}^j + c \delta t_{ru} + K_{ru} + \nu_{\rho ru}^j \\
\rho_{ru}^j &= \rho_{0ru}^j - \hat{\rho}_{0ru}^j \cdot \Delta r_u + \nu_{\rho ru}^j \\
\end{align*}
\]

Or, what is the same:

\[
\begin{align*}
P_{ru}^j - \rho_{0ru}^j &= -\hat{\rho}_{0ru}^j \cdot \Delta r_u + c \delta t_{ru} + K_{ru} + \nu_{\rho ru}^j \\
\end{align*}
\]

Note: for baselines up to 100 km the range error of broadcast orbits is less than 10 cm (assuming \(\epsilon_{eph} \approx 10 \text{ m}\)).

The left hand side of previous equation can be spitted in two terms: one associated to the reference station and the other to the user:

\[
P_{ru}^j - \rho_{0ru}^j = P_{ru}^j - \rho_{0ru}^j - (P_r^j - \rho_{0r}^j)
\]

\[
\Delta r_{ru} = \Delta r_u - \Delta r_r = \Delta r_u - \epsilon_{site}
\]
Differential code based positioning

\[ P^j_{ru} - \rho^j_{o ru} = \hat{\rho}^j_{o u} \cdot \Delta r_u + c \delta t_{ru} + K_{ru} + v^j_{\rho ru} \]

The left hand side of previous equation can be spitted in two terms: one associated to the reference station and the other to the user:

\[ P^j_{ru} - \rho^j_{o ru} = P^j_u - \rho^j_{o u} - (P^j_r - \rho^j_{o r}) \]

- The term \( P^j_{ru} - \rho^j_{o r} \) is the error in range measured by the reference station, which can be broadcasted to the user as a differential correction:

  \[ PRC^j = \rho^j_{o r} - P^j_r \]

- The user applies this differential correction to remove/mitigate common errors:

  \[ P^j_u - \rho^j_{o u} + PRC^j = -\hat{\rho}^j_{o u} \cdot \Delta r_u + c \delta \tilde{t}_{ru} + v^j_{\rho ru} \]

Where the receiver's instrumental delay term \( K_{ru} \) is included in the differential clock \( c \delta \tilde{t}_{ru} \)

For distances beyond a ten of kilometres, or significant altitude difference, it would be preferable to correct for the tropospheric delay at the reference station and user receiver.

Range Differential Correction Calculation

- The **reference station** with known coordinates, computes pseudorange and range-rate corrections: \( PRC = \rho_{ref} - P_{ref} \), \( RRC = \Delta PRC/\Delta t \).

- The **user** receiver applies the PRC and RRC to correct its own measurements, \( P_{user} + (PRC + RRC(t-t_0)) \), removing SIS errors and improving the positioning accuracy.

*DGNSS with code ranges*: users within a hundred of kilometres can obtain **one-meter-level** positioning accuracy using such pseudorange corrections.
Differential code based positioning

The user applies this differential correction to remove/mitigate common errors:

\[
P^j_u - \rho^j_{o_u} + PRC^j = -\hat{\rho}^j_{o_u} \cdot \Delta r_u + c \delta \tilde{t}_{ru} + v^j_{pru}
\]

where the receiver’s instrumental delay term \( K_{ru} \) is included in the differential clock \( c \delta \tilde{t}_{ru} \).

The previous system for navigation equations is written in matrix notation as:

\[
\begin{bmatrix}
\text{Pref}_1^j \\
\text{Pref}_2^j \\
\vdots \\
\text{Pref}_n^j
\end{bmatrix} =
\begin{bmatrix}
-\left(\hat{\rho}^1_{o_u}\right)^T \\
-\left(\hat{\rho}^2_{o_u}\right)^T \\
\vdots \\
-\left(\hat{\rho}^n_{o_u}\right)^T
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
\ddots \\
1
\end{bmatrix}
\begin{bmatrix}
\Delta r_u \\
\Delta \tilde{t}_{ru}
\end{bmatrix}
\]

where

\[
\text{Pref}_j^j \equiv P^j_u - \rho^j_{o_u} + PRC^j
\]

Differential code based positioning

Time synchronization issues:

For simplicity we have dropped any reference to measurement epochs, but real-time implementations entail delays in data transmission and the time update interval can be limited by bandwidth restrictions.

• Differential corrections vary slowly and its useful life can be up to several minutes with S/A=off.

• To reduce bandwidth, the reference station computes Pseudorange Corrections (PRC) and Range-Rate Correction (RRC) for each satellite in view, which are broadcast to every several seconds, up to a minute interval with S/A=off.

• The user computes the PRC at the measurement epoch as:

\[
PRC^j(t) = PRC^j(t_0) + RRC^j(t - t_0)
\]
### Differential code based positioning

#### Data handling:

Reference station and user have to coordinate how the measurements are to be processed:

- Corrections must be identified with an Issue of Data (IOD) and time-out must be considered.
- Both receivers must use the same ephemeris orbits (which are identified by the IODE).
- If reference station uses a tropospheric model the same model must be applied by the user.
- If reference station uses the broadcast ionospheric model, the user must do the same.

**Note:** we have considered here only code measurements. The carrier based positioning will be treated next, using double differences of measurements and targeting the ambiguity fixing.

### Range Differential Correction Calculation

- **Reference station** (known Location)
- **User**

**Broadcast SV Position**

**Actual SV Position**

**Calculated Range** $\rho_{\text{ref}}$

**Measured Pseudoranges**

**Differential Message Broadcast** $\text{PRC, RRC}$

- The **reference station** with known coordinates, computes pseudorange and range-rate corrections: $\text{PRC} = \rho_{\text{ref}} - \rho_{\text{ref}}$, $\text{RRC} = \Delta \text{PRC} / \Delta t$.

- The **user** receiver applies the PRC and RRC to correct its own measurements, $\rho_{\text{user}} + (\text{PRC}(t_0) + \text{RRC}(t-t_0))$, removing SIS errors and improving the positioning accuracy.

**DGNSS with code ranges**: users within a hundred of kilometres can obtain one-meter-level positioning accuracy using such pseudorange corrections.
ftp://cddis.gsfc.nasa.gov/highrate/2013/

1130752.3120 -4831349.1180 3994098.9450 godns
1130760.8760 -4831298.6880 3994155.1860 godns
1112162.1400 -4842853.6280 3985496.0840 usn3

GPS Standalone: 2013 02 21

Vertical Error (GPS Standalone): 2013 02 21

Horizontal Error (GPS Standalone): 2013 02 21

DGPS: reference station

S/A=off
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Introduction: What Augmentation is?

• To enhance the performance of the current GNSS with additional information to:
  - Improve INTEGRITY via real-time monitoring
  - Improve ACCURACY via differential corrections
  - Improve AVAILABILITY and CONTINUITY

• Satellite Based Augmentation Systems (SBAS)
  - E.g., WAAS, EGNOS, MSAS

• Ground Based Augmentation Systems (GBAS)
  - E.g., LAAS

• Aircraft Based Augmentation (ABAS)
  - E.g., RAIM, Inertials, Baro Altimeter

Why Augmentation Systems?

• Current GPS/GLONASS Navigation Systems cannot meet the Requirements for All Phases of Flight:
  - Accuracy
  - Integrity
  - Continuity
  - Availability

• Marine and land users also require some sort of augmentation for improving the GPS/ GLONASS performances.
WHY GNSS NEEDS AN AUGMENTATION?

PERFORMANCE

<table>
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<tr>
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<th>GPS Only</th>
<th>Civil Aviation</th>
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<td><strong>PERFORMANCE</strong></td>
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<td></td>
<td><strong>CATEGORY I</strong></td>
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<tr>
<td>Requirements</td>
<td><strong>ACCURACY (95%)</strong></td>
<td></td>
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<td></td>
<td><strong>INTEGRITY</strong></td>
<td></td>
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<tr>
<td></td>
<td><strong>AVAILABILITY</strong></td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>CONTINUITY OF SERVICE</strong></td>
<td></td>
</tr>
<tr>
<td>H. 13 m</td>
<td>V. 22m</td>
<td>H 16.0 m   V 4.0 m</td>
</tr>
<tr>
<td>99% (RAIM)</td>
<td>99% to 99.990%</td>
<td>2.10^-7/ approach Time to alarm 6 s</td>
</tr>
<tr>
<td>?</td>
<td>INTEGRITY</td>
<td>10^-8 / approach (10^-6 / 15 s)</td>
</tr>
<tr>
<td>?</td>
<td>CONTINUITY OF SERVICE</td>
<td></td>
</tr>
</tbody>
</table>

Even with S/A off a vertical accuracy < 4m 95% of time cannot be guaranteed with the standalone GPS.

Accuracy: Difference between the measured position at any given time to the actual or true position.

ANALYSIS NOTES
- Data taken from Overlook PAN Monitor Station, equipped with Trimble SVeeSix Receiver
- Single Frequency Civil Receiver
- Four Satellite Position Solution at Surveyed Benchmark
- Data presented is raw, no smoothing or editing
**Integrity:** Ability of a system to provide timely warnings to users or to shut itself down when it should not be used for navigation.

Standalone GPS and GLONAS Integrity is Not Guaranteed

GPS/GLONASS Satellites:
Time to alarm is from minutes to hours
No indication of quality of service

Health Messages:
GPS up to 2 hours late
GLONASS up to 16 hours late

**Continuity:** Ability of a system to perform its function without (unpredicted) interruptions during the intended operation.

**Availability:** Ability of a system to perform its function at initiation of intended operation. System availability is the percentage of time that accuracy, integrity and continuity requirements are met.

Availability and Continuity Must meet requirements

**Continuity:**
Less than $10^{-5}$ Chance of Aborting a Procedure Once it is Initiated.

**Availability:**
$>99\%$ for every phase of flight (SARPS).
INTEGRITY

Confidence bound

NSE

Alert Limit

Actual path

Path indicated by the Navigation System

Less than $10^{-7}$ probability of true error larger than confidence bound. Time to alarm 6 s

SBAS and GBAS Navigation Modes

SBAS

WAAS

Enroute Oceanic

Enroute Domestic

Terminal

Approach

Surface

LAAS

GBAS
### Civil Aviation Signal-in-Space Performance Requirements

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<thead>
<tr>
<th>Aviation</th>
<th>Accuracy (H) 95%</th>
<th>Accuracy (V) 95%</th>
<th>Alert Limit (H)</th>
<th>Alert Limit (V)</th>
<th>Integrity</th>
<th>Time to alert</th>
<th>Continuity</th>
<th>Availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>ENR</td>
<td>3.7 Km (2.0 NM)</td>
<td>N/A</td>
<td>7400 m</td>
<td>1850 m</td>
<td>1-10^{-7}/h</td>
<td>5 min.</td>
<td>1-10^{-4}/h to 1-10^{-7}/h</td>
<td>0.99 to 0.99999</td>
</tr>
<tr>
<td>TMA</td>
<td>0.74 Km (0.4 NM)</td>
<td>N/A</td>
<td>1850 m</td>
<td>N/A</td>
<td>1-10^{-7}/h</td>
<td>15 s</td>
<td>1-10^{-4}/h to 1-10^{-7}/h</td>
<td>0.9999 to 0.99999</td>
</tr>
<tr>
<td>NPA</td>
<td>220 m (720 ft)</td>
<td>N/A</td>
<td>600 m</td>
<td>N/A</td>
<td>1-10^{-7}/h</td>
<td>10 s</td>
<td>1-10^{-4}/h to 1-10^{-8}/h</td>
<td>0.99 to 0.99999</td>
</tr>
<tr>
<td>APV-I</td>
<td>220 m (720 ft)</td>
<td>20 m (66 ft)</td>
<td>600 m</td>
<td>50 m</td>
<td>1-2x10^{-7} per approach</td>
<td>10 s</td>
<td>1-8x10^{-6} in any 15 s</td>
<td>0.99 to 0.99999</td>
</tr>
<tr>
<td>APV-II</td>
<td>16.0 m (52 ft)</td>
<td>8.0 m (26 ft)</td>
<td>40 m</td>
<td>20 m</td>
<td>1-2x10^{-7} per approach</td>
<td>6 s</td>
<td>1-8x10^{-6} in any 15 s</td>
<td>0.99 to 0.99999</td>
</tr>
<tr>
<td>CAT-I</td>
<td>16.0 m (52 ft)</td>
<td>6.0 - 4.0 m (20 to 13 ft)</td>
<td>40 m</td>
<td>15 -10 m</td>
<td>1-2x10^{-7} per approach</td>
<td>6 s</td>
<td>1-8x10^{-6} in any 15 s</td>
<td>0.99 to 0.99999</td>
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Most of the measurement errors are common: clock, ephemeris, ionosphere and troposphere.

A common correction valid for any receiver within the LADGPS area is generated and broadcast.

The accuracy is limited by the spatial decorrelation of those error sources (1m at 100Km).

GBAS Concept

The Ground Station (GS) responsible for generating and broadcasting carrier-smoothed code differential corrections and approach-path information to user aircrafts.

This system is used to support aircraft operations during approach and landing.

• It is also responsible of detecting and alarming space-segment and ground-segment failures.

• The GS must insure that all ranging sources for which GBAS corrections are broadcast are safe to use. If a failure occurs that threatens user safety, the GS must detect and alert users (by not broadcasting corrections for the affected ranging source) within a certain time-to-alert.

This is from [RD-5]
GBAS Components

The GBAS station involves redundant Reference Receivers and antennas, redundant High Frequency Data Broadcast equipment linked to a single antenna.

- The GS tracks, decodes, and monitors GPS satellite information and generates differential corrections.
- It also performs integrity checks on the generated corrections.
- The correction message, along with suitable integrity parameters and approach path information, is then broadcast to airborne users on a VHF channel, up to about 40km.

This is from [RD-5]

Ground System Functional Flow Diagram

A variety of integrity monitor algorithms that are grouped into:

- Signal Quality Monitoring (SQM)
- Data Quality Monitoring (DQM)
- Measur. Quality Monitoring (MQM)
- Multiple Reference Consist Check (MRCC)
- $\sigma_M$-monitor
- Message Field Range Test (MFRT)

This is from [RD-5]
**SQM:** Targets satellite signals anomalies and local interference. Implements tests for Correlation Peak Symmetry, Receiver Signal Power and Code-Carrier Divergence.

**DQM:** Checks the validity of the GPS ephemeris and clock data for each satellite that rises in view of the LGF and at each time new navigation data messages are broadcast.

**MQM:** Confirms the consistency of the pseudorange and carrier-phase measurements over the last few epochs to detect sudden step and any other rapid errors.

**MRCC:** Examines the consistency of corrections for each satellite across all reference receivers.

**σμ-monitor:** Helps ensure a Gaussian distribution for the correction error with zero mean and that the broadcast $\sigma_{pr\_gnd\_overbound}$ over bounds the actual errors in the broadcast differential corrections.

**MFRT:** Verifies that the computed averaged pseudorange corrections and correction rates fit within the message field bounds.

This is from [RD-5]

**Executive Monitors (EXM-I and EXM-II) coordinate all previous monitors and combine failure flags.**

- Signal Quality Monitoring (SQM)
- Data Quality Monitoring (DQM)
- Measur. Quality Monitoring (MQM)
- Multiple Reference Consist Check (MRCC)
- Message Field Range Test (MFRT)
1. Linear model for DGNSS: Single Differences
   1.1. Linear model
   1.2. Geographic decorrelation of ephemeris errors
   1.3. Error mitigation and `short´ baseline concept
   1.4. Differential code based positioning

2. Augmentation Systems
   2.1. Introduction
   2.2. Ground-Based Augmentation system (GBAS)
   2.3. Satellite based Augmentation system (SBAS)

---

The pseudorange error is split in its components.
- Clock error
- Ephemeris error
- Ionospheric error
- Local errors (troposphere, multipath, receiver noise)

Uses a network of receivers to cover broad geographic area

Courtesy: FAA
## Error Mitigation

<table>
<thead>
<tr>
<th>Error component</th>
<th>GBAS</th>
<th>SBAS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Satellite clock</td>
<td>Common Mode</td>
<td>Estimation and Removal each error component</td>
</tr>
<tr>
<td>Ephemeris</td>
<td>Differencing</td>
<td></td>
</tr>
<tr>
<td>Ionosphere</td>
<td></td>
<td>Fixed Model</td>
</tr>
<tr>
<td>Troposphere</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Multipath and Receiver Noise</td>
<td>Carrier Smoothing by user</td>
<td></td>
</tr>
</tbody>
</table>

### Three existing SBAS Systems

![Map showing the coverage areas of WAAS, EGNOS, and MSAS](https://example.com/map.jpg)
• EGNOS is the European component of a Satellite Based Augmentation to GPS.

• EGNOS is being developed under the responsibility of a tripartite group:
  - The European Space Agency (ESA)
  - The European Organization for the Safety of Air Navigation (EUROCONTROL)
  - The Commission of the European Union.

ECAC Area
(ECAC: European Civil Aviation Conference)
EGNOS Architecture

GEO (x3)

NLES (x 6)

GPS

RIMS (x 37)

EWAN

MCC 1 MCC 2 MCC 3 MCC 4 PACF ASQF

EGNOS

- MCC
- NLES
- RIMS

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@ J. Sanz & J.M. Juan
**EGNOS ground segment** is composed of the following stations/centres which are mainly distributed in Europe and are interconnected between themselves **through a land network**.

- **37 RIMS** (Ranging and Integrity Monitoring Stations) + seven being deployed: receive the satellite signals and send this information to the MCC centres.

- **4 MCC** (Master Control Centres) receive the information from the RIMS stations and generate correction messages to improve satellite signal accuracy and information messages on the status of the satellites (integrity). The MCC acts as the EGNOS system 'brain'.

- **6 NLES** (Navigation Land Earth Stations): they receive the correction messages from the CPFs for the upload of the data stream to the geostationary satellites and the generation of the GPS-like signal. This data is then transmitted to the European users via the geostationary Satellite

[http://egnos-user-support.essp-sas.eu/egnos_ops/egnos_system/system_description/current_architecture](http://egnos-user-support.essp-sas.eu/egnos_ops/egnos_system/system_description/current_architecture)

---

**Master Control Center (MCC)**

**MCC is Subdivided into**

- **CCF** (Central Control Facility)
  - Monitoring and control EGNOS G/S
  - Mission Monitoring and archive
  - ATC I/F

- **CPF** (Central Processing Facility)
  - Provides EGNOS WAD corrections
  - Ensures the Integrity of the EGNOS users
  - Utilises independent RIMS channels for checking of corrections
  - Real time software system developed to high software standards

**4 MCCs are implemented in EGNOS**
SBAS Differential Corrections and Integrity: The RTCA/ MOPS-DO 229C

Message Format

- 250 bits
- One Message per second
- All messages have identical format

The corrections, even for individual satellites are distributed across several individual messages.
SBAS Broadcast Messages (ICAO SARPS)

<table>
<thead>
<tr>
<th>MSG</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSG 0</td>
<td>Don't use this SBAS signal for anything (for SBAS testing)</td>
</tr>
<tr>
<td>MSG 1</td>
<td>PRN Mask assignments, set up to 51 of 210 bits</td>
</tr>
<tr>
<td>MSG 2 to 5</td>
<td>Fast corrections</td>
</tr>
<tr>
<td>MSG 6</td>
<td>Integrity information</td>
</tr>
<tr>
<td>MSG 7</td>
<td>Fast correction degradation factor</td>
</tr>
<tr>
<td>MSG 8</td>
<td>Reserved for future messages</td>
</tr>
<tr>
<td>MSG 9</td>
<td>GEO navigation message (X, Y, Z, time, etc.)</td>
</tr>
<tr>
<td>MSG 10</td>
<td>Degradation Parameters</td>
</tr>
<tr>
<td>MSG 11</td>
<td>Reserved for future messages</td>
</tr>
<tr>
<td>MSG 12</td>
<td>SBAS Network Time/UTC offset parameters</td>
</tr>
<tr>
<td>MSG 13 to 16</td>
<td>Reserved for future messages</td>
</tr>
<tr>
<td>MSG 17</td>
<td>GEO satellite almanacs</td>
</tr>
<tr>
<td>MSG 18</td>
<td>Ionospheric grid point masks</td>
</tr>
<tr>
<td>MSG 19 to 23</td>
<td>Reserved for future messages</td>
</tr>
<tr>
<td>MSG 24</td>
<td>Mixed fast corrections/long term satellite error corrections</td>
</tr>
<tr>
<td>MSG 25</td>
<td>Long term satellite error corrections</td>
</tr>
<tr>
<td>MSG 26</td>
<td>Ionospheric delay corrections</td>
</tr>
<tr>
<td>MSG 27</td>
<td>SBAS outside service volume degradation</td>
</tr>
<tr>
<td>MSG 28 to 61</td>
<td>Reserved for future messages</td>
</tr>
<tr>
<td>MSG 62</td>
<td>Internal Test Message</td>
</tr>
<tr>
<td>MSG 63</td>
<td>Null Message</td>
</tr>
</tbody>
</table>

Many Message Types Coordinated Through Issues Data (IOD)

Issues of Data (IOD)
Message Time-Outs:

Users can operate even when missing Messages

- Prevents Use of Very Old Data
- Confidence Degrades When Data is Lost
- IODF: Detect Missing Fast Corrections

**Diagram:**
- System Latency: 1 second
- tof: Time of applicability (1st bit of message)
- Last bit of message: tof + 1sec
- The Correction is estimated by the master station

**Table:**

<table>
<thead>
<tr>
<th>Data</th>
<th>Associated Message Types</th>
<th>Maximum Update Interval (seconds)</th>
<th>En Route, Terminal, NPA Timeout (seconds)</th>
<th>Precision Approach Timeout (seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>WAAS in Test Mode</td>
<td>0</td>
<td>6</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>PRN Mask</td>
<td>1</td>
<td>60</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>UDREI</td>
<td>2-6, 24</td>
<td>6</td>
<td>18</td>
<td>12</td>
</tr>
<tr>
<td>Fast Corrections</td>
<td>2-5, 24</td>
<td>60</td>
<td>(*)</td>
<td>(*)</td>
</tr>
<tr>
<td>Long Term Corrections</td>
<td>24, 25</td>
<td>120</td>
<td>360</td>
<td>240</td>
</tr>
<tr>
<td>GEO Nav. Data</td>
<td>9</td>
<td>120</td>
<td>360</td>
<td>240</td>
</tr>
<tr>
<td>Fast Correction Degradation</td>
<td>7</td>
<td>120</td>
<td>360</td>
<td>240</td>
</tr>
<tr>
<td>Weighting Factors</td>
<td>8</td>
<td>120</td>
<td>240</td>
<td>240</td>
</tr>
<tr>
<td>Degradation Parameters</td>
<td>10</td>
<td>120</td>
<td>360</td>
<td>240</td>
</tr>
<tr>
<td>Ionospheric Grid Mask</td>
<td>18</td>
<td>300</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>Ionospheric Corrections</td>
<td>26</td>
<td>300</td>
<td>600</td>
<td>600</td>
</tr>
<tr>
<td>UTC Timing Data</td>
<td>12</td>
<td>300</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>Almanac Data</td>
<td>17</td>
<td>300</td>
<td>None</td>
<td>None</td>
</tr>
</tbody>
</table>

(* Fast Correction Time-Out intervals are given in MT7 [between 12 to 120 sec]
**PRN MASK (MT01)**

<table>
<thead>
<tr>
<th>Bit No</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>.</th>
<th>38</th>
<th>.</th>
<th>120</th>
<th>.</th>
<th>210</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>.</td>
<td>1</td>
<td>.</td>
<td>1</td>
<td>.</td>
<td>0</td>
</tr>
<tr>
<td>PRN</td>
<td>GPS PRN 2</td>
<td>GPS PRN 4</td>
<td>GPS PRN 5</td>
<td>GLONASS Slot 1</td>
<td>AORE PRN 120</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PRN mask Number</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>21</td>
<td>29</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Each MT01 contains its associated IODP

Up to 51 satellites in 210 slots.

Note: Each Correction set in MT 2-5,5,6,7,24,25 its characterized by its PRN-Mask number, between 1 to 51.

---

**Example of message: Fast Corrections (MT2-5,24)**

- **Primarily Removed SA**
  - Common to **ALL** users
  - Up to **13** Satellites Per Message
  - Pseudorange Correction /confidence Bound
  - Range Rate Formed by Differencing
  - UDRE degrades Over Time
    - Acceleration Term in MT 7
    - Reset when new Message Received
Example of message: Fast Corrections (MT2-5,24)

\[ PRC(t) = PRC_n + RRC_n(t - t_n) \]
\[ RRC_n = \frac{PRC_n - PRC_o}{t_n - t_o} \]

\[ Y = C1 + PRC - \rho^* + \Delta t^{sat} + dt^{sat} - TGD + IONO + TROP \]

Example of message: Ionospheric Corrections (MT26)

- Only Required for Precision Approach
  - Grid of Vertical Ionospheric Corrections
  - Users Select 3 or 4 IGPs that Surrounding IPP
    - 5°x5° or 10°x10° for 55° < Lat < 55°
    - Only 10°x10° for 55° < |Lat| < 85°
    - Circular regions for |Lat| > 85°
  - Vertical Correction and UIVE Interpolated to IPP
  - Both Converted to Slant by Obliquity Factor

IGP: Ionospheric Grid Point
IPP: Ionospheric Pierce Point
**Global IGP Grid**

Predefined Global IGP Grid

**Ephemeris + Clocks**

- Fast Corrections
- Long Term Corrections
- UDRE
- Degradation Param.

IONO Corrections
- GIVE
- Degradation Param.

MT1, MT2, 5, 24, MT6, MT25, MT7, MT12, MT9

MT10

MT18

MT26

\[ y = C_1 + \text{PRC} - \rho^* + \Delta t^{sat} + d t^{sat} - \text{TGD} - \text{IONO} - \text{TROP} \]

\[ \sigma^2 = \sigma^2_{flt} + \sigma^2_{UIRE} + \sigma^2_{air} + \sigma^2_{tropo} \]
Navigation System Error and Protection levels

\[ y = G \cdot x \]

\[ \hat{x} = (G^TWG)^{-1} G^TWy \]

\[ P_x = (G^TWG)^{-1} = \begin{bmatrix} d_x^2 & d_{NE} & d_{NY} & d_{NT} \\ d_{NE} & d_x^2 & d_{EY} & d_{ET} \\ d_{NY} & d_{EY} & d_y^2 & d_{VT} \\ d_{NT} & d_{ET} & d_{VT} & d_v^2 \end{bmatrix} \]

\[ W = P_y^{-1} = \begin{bmatrix} \sigma_i^2 & 0 & 0 & 0 \\ 0 & \sigma_i^2 & 0 & 0 \\ 0 & 0 & \sigma_N^2 & 0 \\ 0 & 0 & 0 & \sigma_N^2 \end{bmatrix} \]

\[ \sigma_i^2 = \sigma_{i,flt}^2 + \sigma_{i,URE}^2 + \sigma_{i,air}^2 + \sigma_{i,tropo}^2 \]

\[ HPL = 6.00 \sqrt{\frac{d_x^2 + d_E^2}{2}} + \sqrt{\left(\frac{d_N^2 - d_E^2}{2}\right)^2 + d_{NE}^2} \]

\[ VPL = 5.33 d_v \]

Fast and Long-Term Correction Degradation

\[ \sigma_{i,flt}^2 = \left( \sigma_{i,UDRE} + \mathcal{E}_{fc} + \mathcal{E}_{rrc} + \mathcal{E}_{rte} + \mathcal{E}_{er} \right)^2, \text{if RSS}_{UDRE} = 0 \quad (MT10) \]

\[ \sigma_{i,UDRE}^2 + \mathcal{E}_{fc}^2 + \mathcal{E}_{rrc}^2 + \mathcal{E}_{rte}^2 + \mathcal{E}_{er}^2, \text{if RSS}_{UDRE} = 1 \quad (MT10) \]

\[ \mathcal{E}_{fc} = a \frac{(t-t_0+t_{lat})^2}{2} \]

\[ \mathcal{E}_{rte} = C_{rte} \text{floor} \left( \frac{t-t_{rc}}{T_{rte,0}} \right) \]

\[ (when \ IODF \neq 3) \]

\[ \mathcal{E}_{rc} = \left\{ \begin{array}{ll} 0 & \text{if } t_0 < t < t_0 + I_{rc,0} \\ C_{rc} & \text{otherwise} \end{array} \right. \]

\[ \mathcal{E}_{er} = \left\{ \begin{array}{ll} 0 & \text{if } t_0 < t < t_0 + I_{er,0} \\ C_{er} & \text{otherwise} \end{array} \right. \]

\[ \text{Neither fast nor long term corrections have time out for precision approach} \]

\[ \text{Otherwise} \]

\[ \mathcal{E}_{rrc} = \left\{ \begin{array}{ll} 0 & \text{if } IODF_{previous} - IODF_{current} \equiv 3 \mod 2 \\ a \frac{t}{4} + \frac{B}{2} (t-t_{rc}) & \text{otherwise} \end{array} \right. \]

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Degradation of Ionospheric Corrections

\[ \sigma^2_{UIRE} = F_{pp}^2 \sigma^2_{UIVE} \]

\[ F_{pp} = \left[ 1 - \left( \frac{R \cos E}{R + h_i} \right)^2 \right]^{1/2} \]

\[ \sigma^2_{UIVE} = \sum_{n=1}^{N} W_n \left( x_{pp}, y_{pp} \right) \sigma^2_{n,ionogrid}, \quad N = 4 \text{ or } 3 \]

\[ \sigma^2_{ionogrid} = \begin{cases} \left( \sigma^2_{GIVE} + \mathcal{E}^2_{iono} \right)^2, & \text{if } RSS_{iono} = 0 \quad (MT10) \\ \sigma^2_{GIVE} + \mathcal{E}^2_{iono}, & \text{if } RSS_{iono} = 1 \quad (MT10) \end{cases} \]

\[ \mathcal{E}_{iono} = C_{iono \_step \_floor} \left( \frac{t - t_{iono}}{T_{iono}} \right) + C_{iono \_ramp} \left( t - t_{iono} \right) \]

Users know the receiver-satellites geometry and can compute bounds on the horizontal and vertical position errors. These bounds are called Protection Levels (HPL and VPL). They provide good confidence (10^-7/hour probability) that the true position is within a bubble around the computed position.

\[ P(VPE>VPL) < 10^{-7} / \text{sample} \]

\[ \sigma^2_i = \sigma^2_{i,flt} + \sigma^2_{i,UIRE} + \sigma^2_{i,air} + \sigma^2_{i,tropo} \]
DGNSS implementations: WADGNSS (SBAS)
**References**


---

**EGNOS Safety of Life Service Definition Document** (Ref: EGN-SDD SoL, V1.0). European Comission.


<table>
<thead>
<tr>
<th>Error sources (1σ)</th>
<th>GPS - Error Size (m)</th>
<th>EGNOS - Error Size (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>GPS SREW</td>
<td>4.0&lt;sup&gt;12&lt;/sup&gt;</td>
<td>2.3</td>
</tr>
<tr>
<td>Ionosphere (UIVD error)</td>
<td>2.0 to 5.0&lt;sup&gt;13&lt;/sup&gt;</td>
<td>0.5</td>
</tr>
<tr>
<td>Troposphere (vertical)</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>GPS Receiver noise</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>GPS Multipath (45° elevation)</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>GPS UERE 5° elevation</td>
<td>7.4 to 15.6</td>
<td>4.2 (after EGNOS corrections)</td>
</tr>
<tr>
<td>GPS UERE 90° elevation</td>
<td>4.5 to 6.4</td>
<td>2.4 (after EGNOS corrections)</td>
</tr>
</tbody>
</table>

<sup>12</sup> GPS Standard Positioning Service Performance Standard [RD-3].

<sup>13</sup> This is the typical range of ionospheric residual errors after application of the baseline Klobuchar model broadcast by GPS for mid-latitude regions.

SREW: Satellite Residual Error for the Worst user location.
UIVD: User Ionospheric Vertical Delay.
UERE: User Equivalent Range Error
Lecture 7
Carrier-based Differential Positioning.
Ambiguity Resolution Techniques

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Web site: http://www.gage.upc.edu

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5 March 2017
1. Linear model for DGNSS: Double Differences
   1.1. Differential Code and carrier based positioning
   1.2. Precise relative Positioning
   1.3. The Role of Geometric Diversity

2. Ambiguity resolution Techniques
   2.1. Resolving ambiguities one at a time
   - Single-frequency measurements
   - Dual-frequency measurements
   - Three-frequency measurements
   2.2. Resolving ambiguities as a set: Search techniques
   - Least-Squares Ambiguity Search Technique
   - LAMBDA Method

Linear model for Differential Positioning

Single difference  \((\bullet)^j_r \equiv \Delta(\bullet)^j_u - (\bullet)^j_r\)

\[ P^j_{ru} = \rho^j_{ru} + c \delta t^j_{ru} + T^j_{ru} + I^j_{ru} + K^j_{ru} + v^j_{ru} \]

\[ L^j_{ru} = \rho^j_{ru} + c \delta t^j_{ru} + T^j_{ru} - I^j_{ru} + \lambda_1 \omega^j_{ru} + \lambda_2 N^j_{ru} + b^j_{ru} + v^j_{ru} \]

Double difference  \((\bullet)^{jk}_{ru} \equiv \nabla \Delta(\bullet)^{jk}_{ru} = (\bullet)^k_{ru} - (\bullet)^j_{ru} = (\bullet)^{jk}_{ru} - (\bullet)^j_{ru} - (\bullet)^j_{ru} \]

\[ P^k_{ru} = \rho^k_{ru} + c \delta t^k_{ru} + T^k_{ru} + I^k_{ru} + K^k_{ru} + v^k_{ru} \]

\[ P^j_{ru} = \rho^j_{ru} + c \delta t^j_{ru} + T^j_{ru} + I^j_{ru} + K^j_{ru} + v^j_{ru} \]

\[ P^{jk}_{ru} = \rho^{jk}_{ru} + T^{jk}_{ru} + I^{jk}_{ru} + v^{jk}_{ru} \]

The same for carrier:

\[ L^{jk}_{ru} = \rho^{jk}_{ru} + T^{jk}_{ru} - I^{jk}_{ru} + \lambda_1 \omega^{jk}_{ru} + \lambda_2 N^{jk}_{ru} + v^{jk}_{ru} \]

Now are cancelled:
- Receiver clock
- Receiver code instrumental delays
- Receiver carrier instrumental delays

\(\Rightarrow\) Carrier ambiguities \(N\) are integer
Single-Difference of measurements (corrected by geometric range!!)

\[ \Delta(L - \rho) \equiv L_{i \text{ru}} - \rho_{i \text{ru}}^\text{sat} = \Delta \delta t_{\text{ru}} + T_{\text{ru}}^j - I_{\text{ru}}^j + \lambda \omega_{\text{ru}}^j + \lambda N_{\text{ru}}^j + b_{\text{ru}} + v_{\text{ru}}^j \]

\[ \Delta(P - \rho) \equiv P_{i \text{ru}}^\text{sat} - \rho_{i \text{ru}}^\text{sat} = \Delta \delta t_{\text{ru}} + T_{\text{ru}}^j + I_{\text{ru}}^j + K_{\text{ru}} + v_{\text{ru}}^j \]

**Dif. Wind-up:** Very small

**Dif. Receiver clock:** Main variations

**Dif. Tropo. and Iono.:** Small variations

---

Double-Difference of measurements (corrected by geometric range!!)

\[ \nabla \Delta L - \nabla \Delta \rho \equiv L_{i \text{ru}} - \rho_{i \text{ru}}^\text{sat} \]

\[ \nabla \Delta P - \nabla \Delta \rho \equiv P_{i \text{ru}}^\text{sat} - \rho_{i \text{ru}}^\text{sat} \]

**Dif. wind-up:** negligible

**Dif. Tropo. and Iono.:** Small variations

---

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Linear model for Differential Positioning

Single difference
\[ (\bullet)_r^j = \Delta(\bullet)_r^j = (\bullet)_u^j - (\bullet)_r^j \]

\[ P_r^j = \rho_{ru}^j + c \delta t_r + T_r^j + I_r^j + K_r^j + \nu_r^j \]
\[ L_r^j = \rho_{ru}^j + c \delta t_r + T_r^j - I_r^j + \lambda \omega_r^j + \lambda N_r^j + b_r^j + \nu_r^j \]

where:
\[ \rho_{ru}^j = \rho_{0ru}^j - \hat{\rho}_{0u}^j \cdot \Delta r_{ru} - \hat{\rho}_{0ru}^j \cdot \varepsilon_{site} + \hat{\rho}_{0ru}^j \cdot \varepsilon_{eph} \]

Double difference
\[ (\bullet)_r^{jk} \equiv \nabla \Delta(\bullet)_{ru}^{jk} = (\bullet)^k_{ru} - (\bullet)^j_{ru} = (\bullet)^k_{ru} - (\bullet)^j_{ru} - \left[ (\bullet)_u^j - (\bullet)_r^j \right] \]

\[ P_r^{jk} = \rho_{ru}^{jk} + T_r^{jk} + I_r^{jk} + \nu_{ru}^{jk} \]
\[ L_r^{jk} = \rho_{ru}^{jk} + T_r^{jk} - I_r^{jk} + \lambda \omega_r^{jk} + \lambda N_r^{jk} + \nu_{ru}^{jk} \]

where:
\[ \rho_{ru}^{jk} = \rho_{0ru}^{jk} - \hat{\rho}_{0u}^{jk} \cdot \Delta r_{ru} - \hat{\rho}_{0ru}^{jk} \cdot \varepsilon_{site} + \hat{\rho}_{0ru}^{jk} \cdot \varepsilon_{eph} - \hat{\rho}_{0ru}^{jk} \cdot \varepsilon_{eph} \]

being:
\[ \rho_{0ru}^{jk} \equiv \rho_{0ru}^k - \rho_{0ru}^j \]
\[ \Delta r_{ru} \equiv \Delta r_u - \Delta r_r \]

Exercise:
Consider the Single Differences of geometric range:
\[ \rho_{ru}^j = \rho_{0ru}^j - \hat{\rho}_{0u}^j \cdot \Delta r_{ru} - \hat{\rho}_{0ru}^j \cdot \varepsilon_{site} + \hat{\rho}_{0ru}^j \cdot \varepsilon_{eph} \]

where \( \rho_{ru}^j = \rho_{u}^j - \rho_{r}^j \)

Show that the Double Differences are given by:
\[ \rho_{ru}^{jk} = \rho_{0ru}^{jk} - \hat{\rho}_{0u}^{jk} \cdot \Delta r_{ru} - \hat{\rho}_{0ru}^{jk} \cdot \varepsilon_{site} + \hat{\rho}_{0ru}^{jk} \cdot \varepsilon_{eph} - \hat{\rho}_{0ru}^{jk} \cdot \varepsilon_{eph} \]

being:
\[ \rho_{0ru}^{jk} \equiv \rho_{0ru}^k - \rho_{0ru}^j \]
\[ \Delta r_{ru} \equiv \Delta r_u - \Delta r_r \]
Linear model for Differential Positioning

\[
(\mathbf{\Delta})_{ru}^{jk} \equiv \nabla \Delta(\mathbf{\Delta})_{ru}^{jk} = (\mathbf{\Delta})_{ru}^{jk} - (\mathbf{\Delta})_{ru}^{jk} = (\mathbf{\Delta})_{ru}^{jk} - \left[ (\mathbf{\Delta})_{ru}^{jk} - (\mathbf{\Delta})_{ru}^{kr} \right]
\]

\[
P_{ru}^{jk} = \rho_{ru}^{jk} + \rho_{ru}^{jk} + \nu_{ru}^{jk} + N_{ru}^{jk} + \nu_{ru}^{jk}
\]

\[
L_{ru}^{jk} = \rho_{ru}^{jk} + \rho_{ru}^{jk} - \nu_{ru}^{jk} + \nu_{ru}^{jk} + \Lambda N_{ru}^{jk} + \nu_{ru}^{jk}
\]

where:

\[
\rho_{ru}^{jk} = \rho_{ru}^{jk} - \hat{\rho}_{ru}^{jk} \Delta r_{ru}^{jk} - \rho_{ru}^{jk} + \hat{\rho}_{ru}^{jk} \nu_{ru}^{jk} - \hat{\rho}_{ru}^{jk} \nu_{ru}^{jk}
\]

For short baselines (e.g. up to 10 km) and if the reference station coordinates are accurately known, we can assume:

\[
T_{ru}^{jk} \approx 0; I_{ru}^{jk} \approx 0; \omega_{ru}^{jk} \approx 0
\]

\[
\hat{\rho}_{0ru}^{j} \cdot \nu_{eph} \approx 0
\]

\[
\nu_{sle} \approx 0 \Rightarrow \Delta r_{ru}^{jk} \approx \Delta r_{ru}
\]

With these simplifications, we have:

\[
P_{ru}^{jk} - \rho_{0ru}^{jk} = P_{ru}^{jk} - \rho_{0u}^{jk} - \left( P_{ru}^{jk} - \rho_{0r}^{jk} \right)
\]

\[
L_{ru}^{jk} - \rho_{0r}^{jk} = -\hat{\rho}_{0u}^{jk} \nu_{r}^{jk} + \hat{\rho}_{0u}^{jk} \nu_{r}^{jk}
\]

Remark: \( P_{ru}^{jk} - \rho_{0ru}^{jk} = P_{ru}^{jk} - \rho_{0u}^{jk} - \left( P_{ru}^{jk} - \rho_{0r}^{jk} \right) \)

Differential code and carrier positioning

As with the SD, the left hand side of previous equations can be spitted in two terms: one associated to the reference station and the other to the user. Then, the differential corrections can be computed for code and carrier as:

\[
PRC_{P}^{jk} = \rho_{0r}^{jk} - P_{ru}^{jk}; \quad PRC_{L}^{jk} = \rho_{0r}^{jk} - L_{ru}^{jk}
\]

• The user applies this differential correction to remove/mitigate common errors:

\[
P_{ru}^{jk} - \rho_{0u}^{jk} + PRC_{P}^{jk} = -\hat{\rho}_{0u}^{jk} \nu_{r}^{jk}
\]

\[
L_{ru}^{jk} - \rho_{0u}^{jk} + PRC_{L}^{jk} = -\hat{\rho}_{0u}^{jk} \nu_{r}^{jk} + L_{ru}^{jk}
\]

Where the carrier ambiguities \( N \) are integer numbers and must be estimated together with the user solution.

For larger distances, the atmospheric propagation effects (troposphere, ionosphere) must be removed with accurate modelling. Wide area users will require also orbit corrections.

Remark: \( P_{ru}^{jk} - \rho_{0ru}^{jk} = P_{ru}^{jk} - \rho_{0u}^{jk} - \left( P_{ru}^{jk} - \rho_{0r}^{jk} \right) \)
**Differential code and carrier positioning**

The user applies this differential correction to remove/mitigate common errors:

\[
P_{u}^{jk} - \rho_{o}^{jk} + PRC_{P}^{jk} = \hat{\rho}_{o}^{jk} \cdot \Delta r_{u}^{j} + \nu_{P}^{jk}
\]

\[
L_{u}^{jk} - \rho_{o}^{jk} + PRC_{L}^{jk} = -\hat{\rho}_{o}^{jk} \cdot \Delta r_{u}^{j} + \lambda N_{ru}^{jk} + \nu_{L}^{jk}
\]

where \( \hat{\rho}_{o}^{jk} = \rho_{o}^{k} - \rho_{o}^{j} \)

The previous system for navigation equations is written in matrix notation as:

\[
\begin{bmatrix}
\text{Pref}_{P} & \text{Pref}_{L} \\
\text{Pref}_{P} & \text{Pref}_{L}
\end{bmatrix} =
\begin{bmatrix}
-(\hat{\rho}_{o}^{1}-\hat{\rho}_{o}^{R}) & 0 & \cdots & 0 \\
-(\hat{\rho}_{o}^{n-1}-\hat{\rho}_{o}^{R}) & 1 & \cdots & 0 \\
-(\hat{\rho}_{o}^{n-1}-\hat{\rho}_{o}^{R}) & 0 & \cdots & 0 \\
-(\hat{\rho}_{o}^{n-1}-\hat{\rho}_{o}^{R}) & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
\Delta r_{u}^{j} \\
\lambda N_{R}^{R,1} \\
\lambda N_{R}^{R,n-1}
\end{bmatrix}
\]

where

\[
\begin{align*}
\text{Pref}_{P}^{R,k} & = P_{u}^{R,k} - \rho_{o}^{R,k} + PRC_{P}^{R,k} \\
\text{Pref}_{L}^{R,k} & = L_{u}^{R,k} - \rho_{o}^{R,k} + PRC_{L}^{R,k}
\end{align*}
\]

**Differential positioning**

- The **reference station** with known coordinates, computes differential corrections:
  \[
  PRC_{P}^{jk} = \rho_{o}^{jk} - \rho_{r}^{jk} ; \quad PRC_{L}^{jk} = \rho_{o}^{jk} - L_{r}^{jk}
  \]

- The **user** receiver applies these corrections to its own measurements to remove SIS errors and improve the positioning accuracy.
Correlations among the DD Measurements

We assume uncorrelated measurements (both code and carrier). Then, the covariance matrix is diagonal:

\[ P_p = \sigma_p^2 I \quad P_L = \sigma_L^2 I \]

where, we can assume: \( \sigma_p \approx 50cm \); \( \sigma_L \approx 5mm \)

Let \( X \) represent the code \( P \) or the Carrier \( L \) measurement.

* The **single difference (SD)** and its covariance matrix can be computed as:

\[ \begin{bmatrix} X^k_{ru} \\ X^j_{ru} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} X^u \\ X^r \\ X^v \\ X^r \end{bmatrix} \]

Thence, if the measurements are uncorrelated, so are they in single differences, but the noise is twice!

\[ P_{X,SD} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_X^2 & 0 & 0 & 0 \\ 0 & \sigma_X^2 & 0 & 0 \\ 0 & 0 & \sigma_X^2 & 0 \\ 0 & 0 & 0 & \sigma_X^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2\sigma_X^2 I \]

Note: The removal of the relative “User”—“Reference-station” common bias (e.g. relative receiver clock) in DD is done at the expense of one observation and the introduction of a correlation between measurements.

Correlations among the DD Measurements

* Now, the **double difference (DD)** and its covariance matrix can be computed as:

\[ \begin{bmatrix} X^k_{ru} \\ X^j_{ru} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} X^k \\ X^j \\ X^k \\ X^j \end{bmatrix} \]

Thence, even if the original measurements are uncorrelated, the double differences are correlated.

\[ P_{X,DD} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2\sigma_X^2 & 0 & 0 & 0 \\ 0 & 2\sigma_X^2 & 0 & 0 \\ 0 & 0 & 2\sigma_X^2 & 0 \\ 0 & 0 & 0 & 2\sigma_X^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ -1 & -1 \\ 1 & 1 \end{bmatrix} = 2\sigma_X^2 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]
Single and double differences of receivers/satellites

Receiver errors affecting both satellites are removed (e.g. Receiver clock)

\[ \nabla \mathbf{R} \equiv \nabla \mathbf{k} - \nabla \mathbf{R} \]

SIS errors affecting both receivers are removed (e.g. Satellite clocks, ...)

\[ \Delta \mathbf{R} \equiv \Delta \mathbf{k} - \Delta \mathbf{R} = \nabla \mathbf{R}_{\text{rov}} - \nabla \mathbf{R}_{\text{ref}} \]

Receiver errors common for all satellites do not affect positioning (as they are assimilated in the receiver clock estimate). Thence:

- Only residual errors in single differences between sat. affect absolute posit.
- Only residual errors in double differences between sat. and receivers affect relative positioning.

When comparing SD and DD one might suggest that in the DD formulation there is even further error reduction, positively influencing the results in positioning. This is however not true, since in the SD case the mean value of unmodelled effects is absorbed by the receiver clock. If the DD correlations are taken into account, the positioning results in both cases are the same. However the DD formulation has the advantage that it allows the direct estimation of the ambiguities.
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Relative Positioning

The following relationship can be obtained from the figure, where we assume that the baseline is shorter than the distance to the satellite by orders of magnitude:

\[ \rho_{ru}^j = \rho_u^j - \rho_r^j = -\hat{\rho}_u^j \cdot r_{ru} \]

Applying the same scheme to a second satellite “k”

\[ \rho_{ru}^k = \rho_u^k - \rho_r^k = -\hat{\rho}_u^k \cdot r_{ru} \]

Thence, the double differences of ranges are:

\[ \rho_{ru}^{jk} = \rho_{ru}^k - \rho_{ru}^j = -\left(\hat{\rho}_u^k - \hat{\rho}_u^j\right) \cdot r_{ru} = -\hat{\rho}_u^{jk} \cdot r_{ru} \]
Relative Positioning

Exercise:
Demonstrate the following relationship between the baseline and the differential range [*]:

\[ \rho_{ru}^{j} = \rho_{u}^{j} - \rho_{r}^{j} = -\left( \frac{2\rho_{u}^{j} + \mathbf{r}_{ru}}{\|\mathbf{p}_{u}^{j}\| + \|\mathbf{p}_{u}^{j} + \mathbf{r}_{ru}\|} \right) \cdot \mathbf{r}_{ru} \]

Comments:
The previous expression can be written as:

\[ \rho_{u}^{j} - \rho_{r}^{j} = -\left( \omega^{j} \hat{e}^{j} \right) \cdot \mathbf{r}_{ru} \quad \text{with} \quad \omega^{j} = \frac{2\rho_{u}^{j} + \mathbf{r}_{ru}}{\|\mathbf{p}_{u}^{j}\| + \|\mathbf{p}_{u}^{j} + \mathbf{r}_{ru}\|} \quad \hat{e}^{j} = \frac{\mathbf{p}_{u}^{j} + \mathbf{r}_{ru}/2}{\|\mathbf{p}_{u}^{j} + \mathbf{r}_{ru}/2\|} \]

\[ \text{• Taking } \mathbf{r}_{ru} = 0 \text{ in } \omega^{j} \text{ and } \hat{e}^{j} \text{ leads to the approximate expression previously found.} \]

\[ \text{• } \omega^{j} \text{ and } \hat{e}^{j} \text{ depend on the baseline } \mathbf{r}_{ru}, \text{ which is the vector to estimate. Nevertheless, it is not very sensitive to changes in such baseline and can be computed iteratively, computing the navigation solution starting from } \mathbf{r}_{ru} = 0. \]

[*] This result is from [RD-7]

Relative Positioning

Solution

Consider the following relations:

\[ \mathbf{r}_{ru} = \mathbf{p}_{r}^{j} - \mathbf{p}_{u}^{j} \]

\[ \hat{e}^{j} = \frac{\mathbf{p}_{u}^{j} + \mathbf{r}_{ru}/2}{\|\mathbf{p}_{u}^{j} + \mathbf{r}_{ru}/2\|} = \frac{\mathbf{p}_{r}^{j} + \mathbf{p}_{u}^{j}}{2\mathbf{p}_{u}^{j} + \mathbf{r}_{ru}} \]

\[ \left( \frac{2\rho_{u}^{j} + \mathbf{r}_{ru}}{\|\mathbf{p}_{u}^{j}\| + \|\mathbf{p}_{u}^{j} + \mathbf{r}_{ru}\|} \right) \cdot \mathbf{r}_{ru} = \|\mathbf{p}_{r}^{j}\| - \|\mathbf{p}_{u}^{j}\| \]

\[ = \left( \|\mathbf{p}_{r}^{j}\| - \|\mathbf{p}_{u}^{j}\| \right) \left( \|\mathbf{p}_{u}^{j} + \mathbf{r}_{ru}\| + \|\mathbf{p}_{u}^{j}\| \right) \]

Then:

\[ \rho_{u}^{j} - \rho_{r}^{j} = -\left( \omega^{j} \hat{e}^{j} \right) \cdot \mathbf{r}_{ru} \]

\[ \text{with: } \omega^{j} = \frac{2\rho_{u}^{j} + \mathbf{r}_{ru}}{\|\mathbf{p}_{u}^{j}\| + \|\mathbf{p}_{u}^{j} + \mathbf{r}_{ru}\|} \quad \hat{e}^{j} = \frac{2\rho_{u}^{j} + \mathbf{r}_{ru}}{\|\mathbf{p}_{u}^{j}\| + \|\mathbf{p}_{u}^{j} + \mathbf{r}_{ru}\|} \]

@ J. Sanz & J.M. Juan
Relative Positioning

Thence, the double differences of ranges are:

\[
\rho_{ru}^{jk} = \rho_{ru}^k - \rho_{ru}^j = - (\hat{\rho}_u^k - \hat{\rho}_u^j) \cdot \mathbf{r}_{ru} = - \hat{\mathbf{r}}_{ru}^{jk} \cdot \mathbf{r}_{ru}
\]

As commented before, for short baselines (e.g. less than 10km), we can assume that ephemeris and propagation errors cancel, thence:

\[
\begin{align*}
P_{ru}^{jk} &= \rho_{ru}^{jk} + T_{ru}^{jk} + I_{ru}^{jk} + \nu_{\rho_{ru}}^{jk} \\
L_{ru}^{jk} &= \rho_{ru}^{jk} + T_{ru}^{jk} - I_{ru}^{jk} + \lambda N_{ru}^{jk} + \nu_{L_{ru}}^{jk}
\end{align*}
\]

\[
\begin{align*}
P_{ru}^{jk} &= \hat{\rho}_u^{jk} \cdot \mathbf{r}_{ru} + \nu_{\rho_{ru}}^{jk} \\
L_{ru}^{jk} &= \hat{\rho}_u^{jk} \cdot \mathbf{r}_{ru} + \lambda N_{ru}^{jk} + \nu_{L_{ru}}^{jk}
\end{align*}
\]

Note that these equations allows a direct estimation of the baseline, without needing an accurate knowledge of the reference station coordinates.

The previous system for navigation equations is written in matrix notation as:

\[
\begin{bmatrix}
P_{ru}^{R_1} \\
L_{ru}^{R_1} \\
\vdots \\
P_{ru}^{R_{n-1}} \\
L_{ru}^{R_{n-1}}
\end{bmatrix}
= 
\begin{bmatrix}
- (\hat{\rho}_u^{R_1} - \hat{\rho}_u^R)^T & 0 & \cdots & 0 \\
- (\hat{\rho}_u^{R_{n-1}} - \hat{\rho}_u^R)^T & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
- (\hat{\rho}_u^{R_{n-1}} - \hat{\rho}_u^R)^T & 0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{r}_{ru} \\
\lambda N_{ru}^{R_1} \\
\vdots \\
\lambda N_{ru}^{R_{n-1}}
\end{bmatrix}
\]

where \( \hat{\rho}_u^{jk} = \hat{\rho}_u^k - \hat{\rho}_u^j \)
Double-Difference of measurements

\[
\nabla \Delta L_1 \equiv L_{\text{sat}}^{\text{ru}}
\]

\[
\nabla \Delta P_1 \equiv P_{\text{sat}}^{\text{ru}}
\]

\[
\nabla \Delta L_4 \equiv L_{\text{ru}}^j = -\mathbf{\rho}_{\text{ru}}^j + T_{\text{ru}}^j - I_{\text{ru}}^j + \lambda \omega_{\text{ru}}^j + \lambda N_{\text{ru}}^j + \nu_{\text{ru}}^j
\]

\[
\nabla \Delta P_4 \equiv P_{\text{ru}}^j = -\mathbf{\rho}_{\text{ru}}^j + T_{\text{ru}}^j + I_{\text{ru}}^j + \nu_{\text{ru}}^j
\]

\[
\mathbf{\rho}_{\text{ru}}^j = -\hat{\mathbf{\rho}}_{\text{ru}}^j \cdot \mathbf{r}_{\text{ru}}
\]

Variation of the Baseline projection over the unit Line-Of-Sight vector

@ J. Sanz & J. M. Juan
In this approach, the reference station broadcast its time-tagged code and carrier measurements, instead of the computed differential corrections.

Thence, the user can form the double differences of its own measurements with those of the reference receiver, satellite by satellite, and estimate its position relative to the reference receiver.

Notice that, the baseline can be estimated without needing an accurate knowledge of reference the station coordinates. Of course, the knowledge of the reference station coordinates would allow the user to compute its absolute coordinates.

Relative Positioning

Time synchronization issues:

There is an important and subtle difference between the previous approach of relative positioning (which does not need to know the reference station coordinates) and the differential positioning approach based on the knowledge of the reference station coordinates.

• The differential corrections vary slowly, and its useful life can be up to several minutes with S/A=off.

• But, the measurements change much faster. The range rate can be up to 800m/s and, thence, a synchronizer error of 1millisecond can lead up to more than 1/2 meter of error.

• As commented before, real-time implementation entails also latencies, that can be up to 2 seconds, thence, a extrapolation technique must be applied to the measurements to reduce error due to latency and epoch mismatch (to <1cm if ambiguities are intended to be fixed).
Receiver: JAVAD TRE_G3TH DELTA3.3.12

Master of Science in GNSS

@ J. Sanz & J.M. Juan

1 ms jump of receiver clock adjust

300Km=1ms
COMMENTS

Real-Time implementation entails delays in data transmission, which can reach up to 1 or 2 s.

- Differential corrections vary slowly and its useful life is of several minutes (S/A=off)
- But, the measurements change much faster:
  - The range rate $\frac{d\rho}{dt}$ can be up to 800 m/s and, therefore, the range can change by more than half a meter in 1 millisecond. Moreover, the receiver clock offset can be up to 1 millisecond (depending on the receiver configuration).

  - Thence, the reference station measurements must be:
    - **Synchronized** to reduce station clock mismatch: station clock can be estimated to within 1 μs $\Rightarrow \epsilon_{dt_{\text{sta}}} < 1\text{mm}$
    - **Extrapolated** to reduce error due to latency: carrier can be extrapolated with error < 1 cm.

\[ RRC = \frac{\Delta PRC}{\Delta t} \]

\[ RRC \sim 1 \text{ cm/s} \]

\[ dL_1 \approx d\rho + dt_{\text{rec}} \]

\[ dt_{\text{rec}} \sim 660 \text{ m/s} \]

\[ d\rho \text{ up to } \sim 800 \text{ m/s} \]

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Let us consider again the problem of relative positioning for short baselines. We have previously found the following equation for DD carrier measurements, assuming short baselines (e.g. < 10km)

\[
L_{ru}^{jk} = -\left(\hat{\rho}_{u}^{k} - \hat{\rho}_{u}^{j}\right) \cdot \mathbf{r}_{ru} + \lambda N_{ru}^{jk} + \nu_{ru}^{jk}
\]

This is from [RD-3]

As the ambiguities are constant along continuous carrier phase arcs, an option could be to take differences on time. Thence, if the user and reference receiver are stationary we can write the “triple differences” as:

\[
\delta L_{ru}^{jk} = -\left(\delta \hat{\rho}_{u}^{k} - \delta \hat{\rho}_{u}^{j}\right) \cdot \mathbf{r}_{ru} + \delta \nu_{ru}^{jk} \Rightarrow \begin{bmatrix}
\delta L_{ru}^{12} \\
\delta L_{ru}^{13} \\
\vdots \\
\delta L_{ru}^{1K}
\end{bmatrix} = \begin{bmatrix}
-\left(\delta \hat{\rho}_{u}^{2} - \delta \hat{\rho}_{u}^{1}\right)^{T} \\
-\left(\delta \hat{\rho}_{u}^{3} - \delta \hat{\rho}_{u}^{1}\right)^{T} \\
\vdots \\
-\left(\delta \hat{\rho}_{u}^{K} - \delta \hat{\rho}_{u}^{1}\right)^{T}
\end{bmatrix} \mathbf{r}_{ru} + \mathbf{v}
\]

where:

\[
\delta L_{ru}^{jk} \equiv L_{ru}^{jk}(t_{2}) - L_{ru}^{jk}(t_{1})
\]

For simplicity, we assign \((j=1)\) to the reference satellite.

The Role of Geometric Diversity: **Triple differences**

Now, we have a “clean” equations system involving only the baseline vector to estimate. But the geometry is very weak (the associated DOP will be large number) and the position estimates will be in general worse than those from double differences.
Let us now consider a simple model for the estimation of the relative position vector from SD carrier measurements, assuming short baselines (e.g. <10km):

\[ L_{ru}^j = \rho_{ru}^j + c \delta t_{ru} + \lambda N_{ru}^j + b_{ru} + \nu_{ru}^j \]

\[ \rho_{ru}^j = \rho_u^j - \rho_r^j = -\hat{\rho}_u^j \cdot r_{ru} \]

where \[ d_{ru} = c \delta t_{ru} + b_{ru} \]

Estimation of position and change in position: the role of Geometric Diversity

This is from [RD-3]

Estimation of changes in baseline vector and clock bias is tied to the geometry matrix at time \( t_1 \). This can be well determined.

Estimation of absolute value of baseline vector is tied to the change in geometry matrix at time \( t_1 \). This would be poor determined if such change is not significant.

\[ d_{ru}(t_0) \text{ cannot be estimated at all!} \]

Previous system can be arranged as:

\[ \begin{bmatrix} L_{ru}^1 \\ L_{ru}^2 \\ \vdots \\ L_{ru}^K \end{bmatrix} = \begin{bmatrix} \hat{\rho}_u^1 \, T \\ \hat{\rho}_u^2 \, T \\ \vdots \\ \hat{\rho}_u^K \, T \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} r_{ru} \\ d_{ru} \end{bmatrix} + \begin{bmatrix} \lambda N_{ru}^1 \\ \lambda N_{ru}^2 \\ \vdots \\ \lambda N_{ru}^K \end{bmatrix} + \nu \]

\[ \delta r_{ru}(t_1) = r_{ru}(t_1) - r_{ru}(t_0) \]

Considering now differences between two epochs \( t_0 \) and \( t_1 \), and assuming no cycle-slips:

\[ L_{ru}(t_1) - L_{ru}(t_0) = G(t_1) \begin{bmatrix} \delta r_{ru}(t_1) \\ \delta d_{ru}(t_1) \end{bmatrix} + (G(t_1) - G(t_0)) \begin{bmatrix} r_{ru}(t_0) \\ d_{ru}(t_0) \end{bmatrix} + \tilde{\nu} \]

\[ G(t_1) - G(t_0) = \begin{bmatrix} -\hat{\rho}_u(t_1) \, T \\ -\hat{\rho}_u(t_0) \, T \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ d_{ru}(t_0) \text{ cannot be estimated at all!} \]
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As a driven problem to study the ambiguity fixing, we will consider the problem of differential positioning in DD for short baselines (e.g., < 10 km). In general we will consider that we have Code and Carrier measurements in different frequencies \( q=1,2,\ldots \), i.e., \( P_1, P_2, L_1, L_2 \).

\[
P_{q,ru}^{jk} = \rho_{ru}^{jk} + T_{ru}^{jk} + I_{q,ru}^{jk} + \nu_{q,ru}^{jk} ; \quad q=1,2,\ldots
\]
\[
L_{q,ru}^{jk} = \rho_{ru}^{jk} + T_{ru}^{jk} - I_{q,ru}^{jk} + \lambda_q \omega_{ru}^{jk} + \lambda N_{q,ru}^{jk} + \nu_{q,ru}^{jk}
\]

To simplify notation, when different frequencies are considered, we will remove the subscript \( "ru" \).

\[
P_q^{jk} = \rho_q^{jk} + \nu_{q}^{jk} ; \quad q=1,2,\ldots
\]
\[
L_q^{jk} = \rho_q^{jk} + \lambda_q N_q^{jk} + \nu_{q}^{jk}
\]

We assume the following measurement errors:
\[
\sigma_{P_q} \approx 0.5 \text{ m} \quad \Rightarrow \quad \sigma_{\rho_q} \approx 1 \text{ m}
\]
\[
\sigma_{L_{q}} \approx 0.5 \text{ cm} \quad \Rightarrow \quad \sigma_{\omega_q} \approx 1 \text{ cm}
\]

As commented before, the ambiguity terms are integer numbers, and we can take benefit of this property to fix such ambiguities applying integer ambiguity resolution techniques.
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Resolving ambiguities one at a Time

A simple trial would be (for instance using L1 and P1):

\[ P_{1}^{jk} = \rho_{1}^{jk} + v_{R1}^{jk} \]

\[ L_{1}^{jk} = \rho_{1}^{jk} + \lambda_{1} N_{1}^{jk} + v_{L1}^{jk} \]

\[ L_{1}^{jk} - P_{1}^{jk} = \lambda_{1} N_{1}^{jk} + v_{L1}^{jk} \rightarrow \]

\[ \hat{N}_{1}^{jk} = \frac{L_{1}^{jk} - P_{1}^{jk}}{\lambda_{1}} \text{ roundoff} \]

\[ \lambda_{1} \approx 20 \text{ cm} \]
\[ \sigma_{L1} \approx 1 \text{ m} \]
\[ \sigma_{\hat{N}1} \approx 1 \text{ cm} \]

Too much error (5 wavelengths)!

Note that, assuming a Gaussian distribution of errors, \( \sigma_{\hat{N}1} \approx 1/2 \) guarantee only the 68% of success

As the ambiguity is constant (between cycle-slips), we would try to reduce uncertainty by averaging the estimate on time, but we will need 100 epochs to reduce noise up to 1/2 (but measurement errors are highly correlated on time!)

Similar results with \( L2, P2 \) measurements
L1-F1 ambiguity fixing: TN01-TN02: 7,100m baseline: 2013 062

L2-F2 ambiguity fixing: TN01-TN02: 7,100m baseline: 2013 062
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Resolving ambiguities one at a Time

**Dual frequency** measurements: wide-laning with the Melbourne-Wübbena combination

\[ P_{N}^{jk} = \frac{f_{1}P_{1}^{jk} + f_{2}P_{2}^{jk}}{f_{1} + f_{2}} = \rho^{jk} + \nu_{x}^{jk} \]

\[ L_{1}^{jk} = \rho^{jk} + \lambda_{1}N_{1}^{jk} + \nu_{L_{1}}^{jk} \]
\[ L_{2}^{jk} = \rho^{jk} + \lambda_{2}N_{2}^{jk} + \nu_{L_{2}}^{jk} \]
\[ L_{W}^{jk} = \frac{f_{1}L_{1}^{jk} - f_{2}L_{2}^{jk}}{f_{1} - f_{2}} = \rho^{jk} + \lambda_{W}N_{W}^{jk} + \nu_{L_{W}}^{jk} \]
\[ L_{W} - P_{N}^{jk} = \lambda_{W}N_{W}^{jk} + \nu_{x}^{jk} \]
\[ \hat{N}_{W}^{jk} = \left[ \frac{L_{W} - P_{N}^{jk}}{\lambda_{W}} \right] \text{roundoff} \]

Fixing \( N_{I} \) (after fixing \( N_{W} \))

\[ L_{1}^{jk} - L_{2}^{jk} = \lambda_{1}N_{1}^{jk} - \lambda_{2}N_{2}^{jk} + \nu_{L_{1} - L_{2}}^{jk} \]
\[ = (\lambda_{1} - \lambda_{2})N_{1}^{jk} + \lambda_{2}N_{2}^{jk} + \nu_{L_{1} - L_{2}}^{jk} \]

\[ \hat{N}_{1}^{jk} = \left[ \frac{L_{1}^{jk} - L_{2}^{jk} - \lambda_{2}\hat{N}_{W}^{jk}}{\lambda_{1} - \lambda_{2}} \right] \text{roundoff} \]

\[ \hat{N}_{2}^{jk} = \hat{N}_{1}^{jk} - \hat{N}_{W}^{jk} \]

\( \lambda_{1} = 19.0 \text{ cm} \)
\( \lambda_{2} = 24.4 \text{ cm} \)
\( \lambda_{1} - \lambda_{2} = 5.4 \text{ cm} \)
\( \sigma_{\nu_{L_{1} - L_{2}}} \approx 1 \text{ cm} \)

\( \hat{N}_{W}^{jk} \approx \frac{1}{\lambda_{1} - \lambda_{2}} \sqrt{2\sigma_{\nu_{L_{1} - L_{2}}}} \approx \frac{1.4 \text{ cm}}{5.4 \text{ cm}} \approx 1/4 \)

Now, with uncorrelated measurements from 10 epochs will reduce noise up to about \( \frac{1}{4} \).
Once the integer ambiguities are known, the carrier phase measurements become unambiguous pseudoranges, accurate at the centimetre level (in DD), or better.

Thence, the estimation of the relative position vector is straightforward following the same approach as with pseudoranges.
Exercises:

1) Consider the wide-lane combination of carrier phase measurements

\[ L_w = \frac{f_1 L_1 - f_2 L_2}{f_1 - f_2} \]

where \( L_w \) is given in length units (i.e. \( L_i = \lambda_i \phi_i \)).

Show that the corresponding wavelength is:

\[ \lambda_w = \frac{c}{f_1 - f_2} \]

Hint:

\[ L_w = \lambda_w \phi_w \; ; \; \phi_w = \phi_1 - \phi_2 \]

2) Assuming \( L_1, L_2 \) uncorrelated measurements with equal noise \( \sigma_L \), show that:

\[ \sigma_{\lambda_w} = \frac{\sqrt{\gamma_{12} + 1}}{\sqrt{\gamma_{12} - 1}} \sigma_L \; ; \; \gamma_{12} = \left( \frac{f_1}{f_2} \right)^2 \]

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Three Frequency measurements:

We still consider the above problem of relative positioning in DD for short baselines (e.g. < 10 km) \( \rightarrow \) Ionosphere, troposphere and wind-up differential errors cancel.

$$N_W = N_1 - N_2 \quad ; \quad N_{EW} = N_2 - N_3$$

With three frequency systems, having two close frequencies it is possible to generate an extra-wide-lane signal to enable the single epoch ambiguity fixing.

$$\hat{N}_{EW} = \left[ \frac{L_{EW} - P_{EN}}{\lambda_{EW}} \right]_{\text{roundoff}}$$

$$\hat{N}_w = \left[ \frac{\lambda_{EW} \hat{N}_{EW} -(L_{EW} - L_w)}{\lambda_w} \right]_{\text{roundoff}}$$

$$\hat{N}_i = \left[ \frac{L_1 - L_2 - \lambda_2 \hat{N}_w}{\lambda_1 - \lambda_2} \right]_{\text{roundoff}}$$

Exercise: Justify the previous expressions for \( \sigma \).
**Exercise:**

Repeat the previous study for the Galileo signals E1, E5b and E5a

<table>
<thead>
<tr>
<th>Galileo</th>
<th>Frequency</th>
<th>Wavelengths</th>
<th>Combinations</th>
</tr>
</thead>
<tbody>
<tr>
<td>E1</td>
<td>154 x 10.23 MHz</td>
<td>(\lambda_E = 0.190) m</td>
<td>(\lambda_H - \lambda_E = 0.058) m</td>
</tr>
<tr>
<td>E5b</td>
<td>118 x 10.23 MHz</td>
<td>(\lambda_E = 0.248) m</td>
<td>(\lambda_H = 0.814) m</td>
</tr>
<tr>
<td>E5a</td>
<td>115 x 10.23 MHz</td>
<td>(\lambda_E = 0.255) m</td>
<td>(\lambda_{EW} = 9.768) m</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\hat{N}_{EW} & = \left[ \frac{L_{EW} - P_{EN}}{\lambda_{EW}} \right]_{\text{roundoff}} \quad \sigma_{\hat{N}_{EW}} \approx \frac{1}{\lambda_{EW}} \sigma_{\hat{P}_{EN}} \approx [ ] \quad \gamma_{12} = \left(\frac{f_1}{f_2}\right)^2 = \left(77/59\right)^2 \\
\hat{N}_{W} & = \left[ \frac{\lambda_{EW} \hat{N}_{EW} - (L_{EW} - L_W)}{\lambda_W} \right]_{\text{roundoff}} \quad \sigma_{\hat{N}_W} \approx \frac{1}{\lambda_W} \sigma_{\hat{L}_{EW}} \approx [ ] \quad \gamma_{23} = \left(\frac{f_2}{f_3}\right)^2 = \left(118/115\right)^2 \\
\hat{N}_1 & = \left[ \frac{L_1 - L_2 - \lambda_2 \hat{N}_W}{\lambda_1 - \lambda_2} \right]_{\text{roundoff}} \quad \sigma_{\hat{N}_1} \approx \frac{1}{\lambda_1 - \lambda_2} \sqrt{2} \sigma_{\eta} \approx [ ] \quad \gamma_{12} = \left(\frac{\eta_1}{\eta_2}\right)^2 = \left(154/118\right)^2
\end{align*}
\]

\[
\begin{align*}
L_1 = \rho + \lambda_1 N_1 + \nu_1 \\
L_2 = \rho + \lambda_2 N_2 + \nu_2 \\
L_{EW} = \rho + \lambda_{EW} N_{EW} + \nu_{EW} \quad P_{EN} = \rho + \nu_{EN}
\end{align*}
\[
\begin{align*}
\sigma_{\eta_1} & \approx \sigma_{\eta_2} \approx 1 \text{ cm} \\
\sigma_{\eta_1} & \approx \sigma_{\eta_2} \approx 1 \text{ m}
\end{align*}
\]

Master of Science in GNSS

@ J. Sanz & J. M. Juan

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Resolving Ambiguities as a set

As a driven problem to study the ambiguity fixing, we will consider problem of differential positioning in DD for short baselines (e.g. < 10 km). To simplify, we will consider only carrier measurements at a single or dual frequency.

\[
\begin{align*}
L_q^2(t_i) &= \rho_{12}^2(t_i) + N_q^{12} + \nu_{L_q}^{12}(t_i) \\
L_q^3(t_i) &= \rho_{13}^3(t_i) + N_q^{13} + \nu_{L_q}^{13}(t_i) \\
\vdots \\
L_q^{K-1}(t_i) &= \rho_{1K-1}^{K-1}(t_i) + N_q^{1K-1} + \nu_{L_q}^{1K-1}(t_i)
\end{align*}
\]

In principle, the estimation of ambiguities in this system is not a big problem if we can wait enough time and the unmodelled errors are not so large.

Linear Model:

\[
\begin{align*}
L_q^{jk}(t_i) &= \rho_{ij}^{jk}(t_i) + N_q^{jk} + \nu_{L_q}^{jk}(t_i) \\
\rho_{ij}^{jk}(t_i) &= \rho_{ij}(t_i) - \hat{\rho}_{ij}(t_i) \cdot \Delta r(t_i)
\end{align*}
\]

We can estimate all parameters (position and ambiguities) as a set by considering the over-dimensioned system of linear equations and solving it by the LS criterion.

\[
y(t_i) = G(t_i) \Delta r(t_i) + \lambda N + v
\]
Resolving Ambiguities as a set

\[
y(t_i) = \mathbf{G}(t_i) \Delta \mathbf{r}(t_i) + \lambda \mathbf{N} + \mathbf{v}(t_i)
\]

For static positioning, considering two epochs (for instance):

\[
\begin{bmatrix}
\mathbf{y}(t_i) \\
\mathbf{y}(t_{i+1})
\end{bmatrix} = \begin{bmatrix}
\mathbf{G}(t_i) \\
\mathbf{G}(t_{i+1})
\end{bmatrix} \Delta \mathbf{r} + \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbf{N} + \begin{bmatrix} \mathbf{v}(t_i) \\
\mathbf{v}(t_{i+1})
\end{bmatrix}
\]

\[\tilde{K} = K - 1\]
\[\tilde{K} = 2(K - 1)\]

In general, mixing several epochs, we will write:

\[\mathbf{y} = \mathbf{G} \Delta \mathbf{r} + \lambda \mathbf{A} \mathbf{N} + \mathbf{v}\]

Using the least-squares criterion, we can look for a real valued 3-vector \(\Delta \mathbf{r}\) and a \(\tilde{K}\)-vector of integers \(\mathbf{N}\) that minimizes the cost function (sum of squared residuals):

\[c(\Delta \mathbf{r}, \mathbf{N}) = \|\mathbf{y} - \mathbf{G} \Delta \mathbf{r} + \lambda \mathbf{A} \mathbf{N}\|\]

Weighted norm can be taken as well

The problem can be easily reformulated for the kinematic case. Kalman filtering can be applied as well.

Resolving Ambiguities as a set

Different strategies can be applied:

- **To Float the ambiguities** (i.e. treating the ambiguities as real numbers).
- **To Search ambiguities** over a limited set of integers to ‘find the best solution’.
- **To solve as an Integer Least-Squares problem**.

For an observation span relatively long, e.g. one hour, the floated ambiguities would typically be very close to integers, and the change in the position solution from the float to the fixed solution should not be large.

As the observation span becomes smaller, ambiguity resolution play a more important role. But very short observation spans implies the risk of wrong ambiguity fixing, which can degrade the position solution significantly.

The performance, is thence measured by:

1. Initialization time
2. Reliability (or, correctness) of the integer estimates
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Search techniques

Strategy:

• Define a volume to be searched
• Set up a grid within this volume
• Define a cost function (e.g. the sum of squared residuals)
• Evaluate the cost function at each grid point
Solution corresponds to the grid point with the lowest value of the cost function

<table>
<thead>
<tr>
<th>Position domain</th>
<th>Ambiguity domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ambiguity Function Method (AFM)</td>
<td>LAMBDA (Teunissen, 1993)</td>
</tr>
<tr>
<td>ARCE</td>
<td>MLAMBDA (Chang et al. 2005)</td>
</tr>
<tr>
<td></td>
<td>OMEGA (Kim and Langley, 2000)</td>
</tr>
<tr>
<td></td>
<td>FASF (Chen and Lachappelle, 1995)</td>
</tr>
<tr>
<td></td>
<td>IP (Xu et al., 1995)</td>
</tr>
</tbody>
</table>
A conceptually simpler approach would consist on:

- Estimate the floated solution \( \hat{\mathbf{N}} \) and its uncertainty (e.g. \( \hat{\mathbf{N}} = 2502347.74 \) cycles, \( \sigma_{\hat{\mathbf{N}}} = 0.6 \) cycles)
- Define as a volume to be searched (e.g. \( \pm 3\sigma_{\hat{\mathbf{N}}} \approx \pm 2 \) cycles) and evaluate the cost function (the RMS residuals) over the 6 ambiguities: 2502345, ..., 2502350

The previous search must be done for each satellite in view.
- If there are 5 satellites tracked \( \Rightarrow 4 \) DD ambiguities \( \Rightarrow 6^4 = 1296 \) combinations
- If there are 8 satellites tracked \( \Rightarrow 7 \) DD ambiguities \( \Rightarrow 6^7 = 279376 \) combinations

The integer ambiguity solution corresponding to the smallest RMS residuals is used to select the candidate. However if two or more candidates give roughly similar values of RMS, the test can not be resolved.

- A ratio test (of 2 or 3, depending of the algorithm) between the two smallest RMS is often used to validate the test.

If the ratio is under these values, no integer solution can be determined and it is better to use the floated solution.

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LAMBDA Method

Consider again the previous problem of estimating $\Delta r$, a 3-vector of real numbers, and $N$ a $(K-1)$-vector of integers, which are solution of:

$$y = G \Delta r + \lambda A N + v$$

$$\min \| y - G \Delta r - \lambda A N \|_{W_y}$$

To better exploit the internal correlations $[*]$, we consider now the covariance $W_y = P_y^{-1}$

Let be the float solution and covariance matrix:

$$\begin{bmatrix} \hat{\Delta r} \\ \hat{N} \end{bmatrix} ; \quad \text{Cov} \begin{bmatrix} \Delta r \\ N \end{bmatrix} = \begin{bmatrix} P_{\Delta r} & P_{\Delta r,N} \\ P_{\Delta r,N} & P_N \end{bmatrix}$$

It can be shown the following orthogonal decomposition:

$$\| y - G \Delta r - \lambda A N \|_{W_y}^2 = \| y - G \hat{\Delta r} - \lambda \hat{N} \|_{W_y}^2 + \| \Delta r - \hat{\Delta r}(N) \|_{W_y(N)}^2 + \lambda^2 \| N - \hat{N} \|_{W_N}^2$$

This term must be minimized over the integers

Thence, we have to find $\Delta r$ a 3-vector of real numbers, and $N$ a $(K-1)$-vector of integers minimizing:

$$\| y - G \Delta r - \lambda A N \|_{W_y}^2 = \| y - G \hat{\Delta r} - \lambda \hat{N} \|_{W_y}^2 + \| \Delta r - \hat{\Delta r}(N) \|_{W_y(N)}^2 + \lambda^2 \| N - \hat{N} \|_{W_N}^2$$

This term is irrelevant for minimization since it does not depend on $\Delta r$ and $N$.

This term can be made zero for any $N$

This term must be minimized over the integers

Float solution and covariance matrix:

$$\begin{bmatrix} \Delta r \\ \hat{N} \end{bmatrix} ; \quad \text{Cov} \begin{bmatrix} \hat{\Delta r} \\ \hat{N} \end{bmatrix} = \begin{bmatrix} P_{\Delta r} & P_{\Delta r,N} \\ P_{\Delta r,N} & P_N \end{bmatrix}$$

$W_N^{-1} = P_N \quad W_{\Delta r,N} = P_{\Delta r,N}^{-1}$

$$\min \| N - \hat{N} \|_{W_N}^2 \rightarrow \hat{N}$$

$\Delta r = \hat{\Delta r}(N) = \hat{\Delta r} - W_{\Delta r,N} W_N^{-1} (N - \hat{N})$

The vectors $\Delta r$ and $\hat{N}$ are often referred to as the fixed user solution and fixed ambiguity.
The integer search: Finding the integer vector $\mathbf{N}$ that minimizes the cost function:

$$c(\mathbf{N}) = \left\| \mathbf{N} - \hat{\mathbf{N}} \right\|_{W_\mathbf{N}} = (\mathbf{N} - \hat{\mathbf{N}})^T W_\mathbf{N} (\mathbf{N} - \hat{\mathbf{N}}),$$

where $W_\mathbf{N} = P_\mathbf{N}^{-1}$.

- A diagonal $W_\mathbf{N}$ matrix would mean that the integer ambiguity estimates are uncorrelated.
- If the weight $W_\mathbf{N}$ matrix is diagonal, the minimizing of the cost function is trivial. The best estimate is the float ambiguity rounded to the nearest integer.

$$W_\mathbf{N} = \begin{bmatrix} 1/\sigma^2_{b_1} & 0 \\ 0 & 1/\sigma^2_{b_2} \end{bmatrix}$$

$$c(\mathbf{N}) = \frac{(N_1 - \hat{N}_1)^2}{\sigma^2_{b_1}} + \frac{(N_2 - \hat{N}_2)^2}{\sigma^2_{b_2}}$$

Ellipse parallel to coordinate axes

In practice, the estimated (float) ambiguities are highly correlated and the ellipsoidal region stretches over a wide range of cycles. This is specially the case when the measurements are limited to a single epoch or only a few epochs.

Thence, points that appear much further away from the floated solution may have lower values of cost function than those which appear nearby. In this context, the search for integer vectors can be extremely inefficient.

- To improve the computational efficiency of the search, the float ambiguities should be transformed so that the elongated ellipsoid turns into a sphere-like. Thus, the search can be limited to the neighbours of the floated ambiguity.

- Moreover, this transformation should make the matrix $W$ as close as possible to diagonal (i.e. decorrelating the ambiguities).

Note: $W$ is a positive definite matrix and thence, can be always diagonalized (as a real-valued matrix) with orthogonal eigenvectors. But the problem here is that the integer ambiguities $\mathbf{N}$ must be transformed preserving its integer nature!

Thence, we are looking for an “integer-valued” transformation matrix $Z$ that makes the matrix $W$ as close as possible to a diagonal matrix and with quite similar axes (spherical).

Moreover, the inverse of transformation matrix $Z^{-1}$ must be also integer, to transform back the results after finding the ambiguities.

Note that $Z, Z^{-1}$ integers $\Rightarrow \det(Z) = 1$ (i.e. it is a volume-preserving transformation)
Exercise:

Show that:

\[ Z, \quad Z^{-1} \text{ Integers } \Rightarrow |\text{det}(Z)| = 1 \]

That is, \( Z \) is a volume-preserving transformation

Decorrelation: Computing the Z-transform

The following conditions must be fulfilled:
1. \( Z \) must have integer entries
2. \( Z \) must be invertible and have integer entries
3. The transformation \( Z \) must reduce the product of all ambiguity variances.

Note that \( Z, \quad Z^{-1} \text{ integers } \Rightarrow |\text{det}(Z)| = 1 \)
(i.e. it is a volume-preserving transformation)

Gauss manipulation over matrix \( P = W^{-1} \) can be applied to find-out the matrix \( Z \).

\[ P = \begin{bmatrix} p_{N_1\hat{N}_1} & p_{N_1\hat{N}_2} \\ p_{N_2\hat{N}_1} & p_{N_2\hat{N}_2} \end{bmatrix} \quad Z_1 = \begin{bmatrix} 1 & 0 \\ \alpha_1 & 1 \end{bmatrix} \quad \rightarrow \text{Transforms } N_2 \ (N_1 \text{ remains unchanged}) \]
\[ \alpha_i = -\text{int} \left[ \frac{p_{N_1\hat{N}_2}}{p_{N_i\hat{N}_i}} \right] \]

\[ Z_2 = \begin{bmatrix} 1 & \alpha_2 \\ 0 & 1 \end{bmatrix} \quad \rightarrow \text{Transforms } N_1 \ (N_2 \text{ remains unchanged}) \]

Note: Inverse matrices have also integer entries

\[ Z_1^{-1} = \begin{bmatrix} 1 & 0 \\ -\alpha_1 & 1 \end{bmatrix} \quad Z_2^{-1} = \begin{bmatrix} 1 & -\alpha_2 \\ 0 & 1 \end{bmatrix} \]

Start transforming first the element with largest variance.
Gauss manipulation over matrix $P = W^{-1}$ can be applied to find out the matrix $Z$

$$P_N = \begin{bmatrix} p_{\tilde{N}_1 \tilde{N}_1} & p_{\tilde{N}_1 \tilde{N}_2} \\ p_{\tilde{N}_2 \tilde{N}_1} & p_{\tilde{N}_2 \tilde{N}_2} \end{bmatrix} \quad Z_1 = \begin{bmatrix} 1 & 0 \\ \alpha_1 & 1 \end{bmatrix} \quad \Rightarrow \text{Transforms } N_2 (N_1 \text{ remains unchanged})$$

$$Z_2 = \begin{bmatrix} 1 & \alpha_2 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \text{Transforms } N_1 (N_2 \text{ remains unchanged})$$

**Example:**

$$\hat{N} = \begin{bmatrix} 1.05 \\ 1.30 \end{bmatrix} \quad P_N = \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix}$$

**Step 1:**

$$Z_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \alpha_2 = - \text{int} \left[ 38.4 / 28.0 \right] = -1$$

$$P_{N'} = Z_2 P_N Z_2^T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4.6 & 10.4 \\ 10.4 & 28.0 \end{bmatrix}$$

We transform first the element with largest variance (in this case $N'_1$)

**Step 2:**

$$Z_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \quad \alpha_1 = - \text{int} \left[ 10.4 / 4.6 \right] = -2$$

$$P_{N''} = Z_1 P_{N'} Z_1^T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4.6 & 10.4 \\ 10.4 & 28.0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4.6 & 1.2 \\ 1.2 & 4.8 \end{bmatrix}$$

In general, to increase the number of small off-diagonal elements, we have to transform first the elements with largest variance

Example and pictures from [RD-4]
Example:

\[
\hat{\mathbf{N}} = \begin{bmatrix} 1.05 \\ 1.30 \end{bmatrix} \quad \mathbf{P}_N = \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix} \quad \mathbf{P}_N'' = \begin{bmatrix} 4.6 & 1.2 \\ 1.2 & 4.8 \end{bmatrix}
\]

\[
\hat{\mathbf{N}}'' = \mathbf{Z} \hat{\mathbf{N}} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1.05 \\ 1.30 \end{bmatrix} = \begin{bmatrix} -0.25 \\ 1.80 \end{bmatrix}
\]

\[
\hat{\mathbf{N}}'' = \text{int} \begin{bmatrix} -0.25 \\ 1.80 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]

\[
\hat{\mathbf{N}} = \mathbf{Z}^T \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}
\]

Let \( \mathbf{P} \) be a symmetric and positive-definite matrix:

\[
\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}
\]

\[
\lambda_1 = \frac{1}{2} (p_{11} + p_{22} + \omega) \quad \lambda_2 = \frac{1}{2} (p_{11} + p_{22} - \omega) \quad \omega = \sqrt{(p_{11} - p_{22})^2 + 4 p_{12}^2}
\]

\[
\tan 2\phi = \frac{2 p_{12}}{p_{11} - p_{22}}
\]

Example:

\[
\mathbf{P}_N = \begin{bmatrix} 53.4 & 38.4 \\ 38.4 & 28.0 \end{bmatrix} \quad \hat{\mathbf{N}} = \begin{bmatrix} 1.05 \\ 1.30 \end{bmatrix}
\]

\[
\sqrt{\lambda_1} = \sqrt{81.14} = 9.0 \quad \sqrt{\lambda_2} = \sqrt{0.25} = 0.5
\]

\[
\tan 2\phi = 3.02 \Rightarrow \phi = 35.85^\circ
\]
Consider again the previous problem of estimating $\Delta \mathbf{r}$, a $3$-vector of real numbers, and $\mathbf{N}$ a $(K-1)$-vector of integers, which are solution of

$$
y = G \Delta \mathbf{r} + \lambda \mathbf{A} \mathbf{N} + \mathbf{v}
$$

The solution comprises the following steps:

1. Obtain the float solution and its covariance matrix:

   $$
   \begin{bmatrix}
   \Delta \hat{\mathbf{r}} \\
   \hat{\mathbf{N}}
   \end{bmatrix}
   ,
   \begin{bmatrix}
   \mathbf{P}_{\Delta \mathbf{r}} & \mathbf{P}_{\Delta \mathbf{r}, \mathbf{N}} \\
   \mathbf{P}_{\Delta \mathbf{r}, \mathbf{N}}^T & \mathbf{P}_{\mathbf{N}}
   \end{bmatrix}
   $$

2. Find the integer vector $\mathbf{N}$ which minimizes the cost function

   $$
c(\mathbf{N}) = \|\mathbf{N} - \hat{\mathbf{N}}\|_W^2 = (\mathbf{N} - \hat{\mathbf{N}})^T \mathbf{W}_N (\mathbf{N} - \hat{\mathbf{N}})
   $$

   $\mathbf{W}_N = \mathbf{P}_N^{-1}$

   a) **Decorrelation**: Using the $Z$ transform, the ambiguity search space is re-parametrized to decorrelate the float ambiguities.

   b) **Integer ambiguities estimation** (e.g. using sequential conditional least-squares adjustment, together with a discrete search strategy).

   c) Using the $Z^{-1}$ transform, the ambiguities are transformed to the original ambiguity space.

3. Obtain the ‘fixed’ solution $\Delta \mathbf{r}$, from the fixed ambiguities $\mathbf{N}$.

   $$
y - \lambda \mathbf{A} \mathbf{N} = G \Delta \mathbf{r} + \mathbf{v}
   $$

### b) Integer ambiguities estimation

Several approach can be applied:

- Integer rounding
- Integer bootstrapping
- Integer Least-Squares
-.....

**Comment:** In principle, the previous transformation $Z$ is not required by the estimation concept; it is only to achieve considerable gain in speed in the computation process [RD-5].

### b1) Integer rounding

This is the simplest way.

Just to round-up the ambiguity vector entries to its nearest integer

$$
\hat{\mathbf{N}} = \left(\text{int}(\hat{N}_1), \ldots, \text{int}(\hat{N}_K)\right)
$$

For instance, in the previous example:

$$
\hat{N}'' = \text{int} \begin{bmatrix} -0.25 \\ 1.80 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}
$$
b2) Integer bootstrapping (from [RD-6])

It makes use of integer rounding, but it takes some of the correlations between the ambiguities into account.

1. We start with the most precise floated ambiguity (here we will assume $\hat{N}_{n}$).
2. Then, the remaining float ambiguities are corrected taking into account their correlation with the last ambiguity.

\[
\hat{N}_{n} = \text{int}\left[ \hat{N}_{n} \right] \\
\hat{N}_{n-1} = \text{int}\left[ \hat{N}_{n-1/n} \right] = \text{int}\left[ \hat{N}_{n-1} - \sigma_{\hat{N}_{n-1},\hat{N}_{n}} \sigma_{\hat{N}_{n}}^{2} \left( \hat{N}_{n} - \hat{N}_{n} \right) \right] \\
\vdots \\
\hat{N}_{1} = \text{int}\left[ \hat{N}_{n/f} \right] = \text{int}\left[ \hat{N}_{1} - \sum_{i=2}^{n} \sigma_{\hat{N}_{i},\hat{N}_{i}} \sigma_{\hat{N}_{i}}^{2} \left( \hat{N}_{i} - \hat{N}_{i} \right) \right]
\]

$\hat{N}_{i/l}$ Stands for the $i$-th ambiguity obtained through a conditioning of the previous $I = \{i+1, \ldots, n\}$ sequentially rounded ambiguities.

Using the triangular decomposition

\[
P_{N} = L^{T} DL \\
l_{ij} = \sigma_{\hat{N}_{i},\hat{N}_{i}} \sigma_{\hat{N}_{i}}^{-2}
\]

Example and pictures from [RD-6]

\[
N_{1} = \hat{N}_{1} - \sigma_{\hat{N}_{1},\hat{N}_{1}} \sigma_{\hat{N}_{1}}^{-2} (\hat{N}_{2} - \hat{N}_{2}) \\
\hat{N}_{2} = \text{nint}\left[ \hat{N}_{2} \right] = 0 \\
N_{1} = \text{nint}\left[ \hat{N}_{1/l} \right] = \text{nint}\left[ \hat{N}_{1} - \sigma_{\hat{N}_{1},\hat{N}_{1}} \sigma_{\hat{N}_{1}}^{-2} (\hat{N}_{2} - \hat{N}_{2}) \right] = 1
\]

Figure 2.2: Principle and 2D pull-in regions for integer bootstrapping: parallelograms.
b3) Integer Least Squares (ISL) (from [RD-6])

1. The target is to find the integer vector \( \mathbf{N} \) which minimizes the cost function

\[
  c(\mathbf{N}) = \|\mathbf{N} - \mathbf{\hat{N}}\|^{-1} = \left(\mathbf{N} - \mathbf{\hat{N}}\right)^T \mathbf{P}_N^{-1} (\mathbf{N} - \mathbf{\hat{N}})
\]

\[
  \mathbf{W}_{\mathbf{N}} = \mathbf{P}_N^{-1}
\]

2. The integer minimiser is obtained through a search over the integer grid points on the \( n \)-dimensional hyper-ellipsoid:

\[
  (\mathbf{N} - \mathbf{\hat{N}})^T \mathbf{P}_N^{-1} (\mathbf{N} - \mathbf{\hat{N}}) \leq \chi^2
\]

- Where \( \chi^2 \) determines the size of search region.
- The solution is the integer grid point \( \mathbf{N} \), inside the ellipsoid, giving the minimum value of cost function \( c(\mathbf{N}) \).

Using the triangular decomposition:

\[
  \mathbf{P}_N = \mathbf{L}^T \mathbf{D} \mathbf{L}
\]

\[
  (\mathbf{N} - \mathbf{\hat{N}})^T \mathbf{L}^{-1} \mathbf{D}^{-1} \mathbf{L}^T (\mathbf{N} - \mathbf{\hat{N}}) \leq \chi^2
\]

Defining:

\[
  \mathbf{\tilde{N}} = \mathbf{N} - \mathbf{L}^T (\mathbf{N} - \mathbf{\hat{N}}) \rightarrow \mathbf{L}^T (\mathbf{\tilde{N}} - \mathbf{N}) = (\mathbf{\tilde{N}} - \mathbf{N})
\]

But \( \mathbf{\tilde{N}}_i \) depends on \( \mathbf{\tilde{N}}_{i+1}, \ldots, \mathbf{\tilde{N}}_n \).

\[
  \mathbf{\tilde{N}}_n = \mathbf{\tilde{N}}_n \rightarrow \mathbf{\tilde{N}}_i = \mathbf{\tilde{N}}_i + \sum_{j=i+1}^{n} (\mathbf{N}_j - \mathbf{\tilde{N}}_i) l_{ij}, \quad i = n-1, n-2, \ldots, 1
\]

**Search region bounds:**

\[
  \mathbf{\tilde{N}}_n - d_n^{1/2} \chi < \mathbf{\tilde{N}}_n < \mathbf{\tilde{N}}_n + d_n^{1/2} \chi
\]

\[
  \mathbf{\tilde{N}}_{n-1} - d_{n-1}^{1/2} \left( \chi^2 - (\mathbf{N}_n - \mathbf{\tilde{N}}_n)^2 d_n \right)^{1/2} \leq \mathbf{\tilde{N}}_{n-1} < \mathbf{\tilde{N}}_{n-1} + d_{n-1}^{1/2} \left( \chi^2 - (\mathbf{N}_n - \mathbf{\tilde{N}}_n)^2 d_n \right)^{1/2}
\]

\[
  \vdots
\]

\[
  \mathbf{\tilde{N}}_1 - d_1^{1/2} \left( \chi^2 - \sum_{j=2}^{n} (\mathbf{N}_j - \mathbf{\tilde{N}}_j)^2 d_j \right)^{1/2} < \mathbf{\tilde{N}}_1 < \mathbf{\tilde{N}}_1 + d_1^{1/2} \left( \chi^2 - \sum_{j=2}^{n} (\mathbf{N}_j - \mathbf{\tilde{N}}_j)^2 d_j \right)^{1/2}
\]

**Acceptance test:** The integer ambiguity solution corresponding to the smallest RMS residuals is used to select the candidate. However, if two or more candidates give roughly similar values of RMS, the test cannot be resolved.

A ratio test (of 2 or 3, depending on the algorithm) between the two smallest RMS is often used to validate the test.
Ellipsoid size: selecting the candidates for the acceptance test

The size of the ellipsoidal search region \((N - \hat{N})^T D^{-1} (N - \hat{N}) \leq \chi^2\) is controlled by \(\chi^2\).

Therefore, the performance of the search process is highly dependent on \(\chi^2\):

- A small \(\chi^2\) may result in an ellipsoidal region that fails to contain the solution.
- A too large value for \(\chi^2\) may result in high time-consumming for the search process.

Search with enumeration: When the number of required candidates is at most \(n+1\) (with \(n=\text{dim}(N)\)), the following procedure can be applied to set the value \(\chi^2\):

- The best determined ambiguity is rounded to its nearest integer. The remaining ambiguities are then rounded using their correlations with the first ambiguity:

\[
\hat{N}_n = \text{nint}\left[ \frac{N_n}{\sum_{i=2}^{n} \sigma_{N_i}\sigma_{N_n}^2 (N_n - N_i)} \right]
\]

\[
\hat{N}_{n-1} = \text{nint}\left[ \frac{N_{n-1}}{\sum_{i=2}^{n} \sigma_{N_i}\sigma_{N_n}^2 (N_n - N_i)} \right] = \text{nint}\left[ \frac{N_{n-1} - \sigma_{N_n} \sigma_{N_n}^2 (N_n - N_n)}{\sum_{i=2}^{n} \sigma_{N_i}\sigma_{N_n}^2 (N_n - N_i)} \right] = \text{nint}\left[ \frac{N_{n-1}}{\sum_{i=2}^{n} \sigma_{N_i}\sigma_{N_n}^2 (N_n - N_i)} \right]
\]

...and so on...

- In each step of the conditional rounding procedure, two candidates are taken: The nearest and second-nearest, and conditional rounding is proceeded in both cases.
- If \(p\) candidates are requested, the values of cost function \(c(N)\) are ordered in ascending order and \(\chi^2\) is chosen equal to the \(p\)-th value.

If more than \(n+1\) candidates are requested, the volume of the search ellipsoid can be used ([RD-6]).

Search with shrinking technique: practical example

This is an alternative to the previous strategy, based on shrinking the search ellipsoid during the process of finding the candidates.

In the next example, we have to choose 6 candidates:

- Fist candidate is the bootstrapped solution
- Round “Ambiguity-2” to the second near integer and round the new conditional estimate for “Ambiguity-1”
- The other 5 candidates are found by choosing the conditional “Ambiguity-1” to the 2\textsuperscript{nd}, 3\textsuperscript{rd}, 4\textsuperscript{th} and 5\textsuperscript{th} nearest integers
- The candidate with the largest \(\chi^2\) is removed.

Example and pictures from [RD-6]
Thence, the best 6 candidates are found (in the ISL sense). The one with the smallest cost function $c(N)$ value is the actual ISL solution.

**Acceptance Test**

The integer ambiguity solution corresponding to the smallest RMS residuals is used to select the candidate.

However if two or more candidates give roughly similar values of RMS, the test can not be resolutive.

⇒ A ratio test (of 2 or 3, depending on the algorithm) between the two smallest RMS is often used to validate the test.

If the ratio is under these values, no integer solution can be determined and is better to use the floated solution.

$$RMS = \left\| N - \hat{N} \right\|_{P_k^{-1}} = \sqrt{(N - \hat{N})^T P_k^{-1} (N - \hat{N})}$$
Examples with MATLAB (octave)

load large → Q, a

\[ Q \equiv P_N = W_N^{-1} \]

\[[Q_z, Z_t, L_z, D_z, a_z, i_z] = \text{decorrel}(Q, a);\]

Note:
This document uses the transposed matrix \( Z^T \), but the principle is the same.
[Qz,Zt,Lz,Dz,az,iZ] = decorrel (Q,a);

\[ Z = Zt' \]

\[ Qz = Lz' \cdot \text{diag}(Dz) \cdot Lz \]

\[ a = \text{inv}(Z) \cdot az \]

\[ Q = \text{inv}(Z) \cdot Qz \cdot \text{inv}(Z') \]

\[ [Qz,Zt,Lz,Dz,az,iZ] = \text{decorrel} (Q,a); \]

\[ \text{load large } Q, a \]

**Integer rounding**

\[ \text{round}(a) \]

\[ [ -28491 \ 65753 \ 38830 \ 5004 \ -29196 \ -298 \ -22201 \ 51236 \ 30258 \ 3899 \ -22749 \ -159 ] \]

**Decorrelation + Integer rounding**

\[ [Qz,Zt,Lz,Dz,az,iZ] = \text{decorrel} (Q,a) \]

\[ \text{azfixed}=\text{round}(az); \]

\[ \text{afixed}=iZ*\text{azfixed} \]

\[ [ -28537 \ 65473 \ 38692 \ 4939 \ -29228 \ -504 \ -22237 \ 51018 \ 30150 \ 3849 \ -22774 \ -320 ] \]

**Decorrelation + bootstrapping**

\[ [Qz,Zt,Lz,Dz,az,iZ] = \text{decorrel} (Q,a) \]

\[ \text{azfixed} = \text{bootstrap}(az,Lz); \]

\[ \text{afixed}=iZ*\text{azfixed} \]

\[ [ -28451 \ 65749 \ 38814 \ 5025 \ -29165 \ -278 \ -22170 \ 51233 \ 30245 \ 3916 \ -22725 \ -144 ] \]
Decorrelation + bootstrapping

\[ [Qz,Zt,Lz,Dz,az,iZ] = \text{decorrel} \ (Q,a) \]
\[ \text{azfixed} = \text{bootstrap}(az,Lz) \]
\[ \text{afixed} = iZ*\text{azfixed} \]

\[
\begin{bmatrix}
-28279 & 65862 & 38805 & 5170 & -29061 & -192 & -22036 & 51321 & 30238 & 4029 & -22644 & -77 \\
-28277 & 65935 & 39032 & 4844 & -29337 & -178 & -22385 & 51378 & 30415 & 3775 & -22859 & 8 \\
-28546 & 66062 & 39027 & 4998 & -29228 & -83 & -2244 & 51477 & 30411 & 3895 & -22774 & 8 \\
\end{bmatrix}
\]

Decorrelation + ILS with enumeration search

\[ [Qz,Zt,Lz,Dz,az,iZ] = \text{decorrel} \ (Q,a) ; \]
\[ [\text{azfixed},\text{sqnorm}] = \text{lsearch} \ (az,Lz,Dz,6) ; \]
\[ \text{afixed} = iZ*\text{azfixed} \]

\[
\begin{bmatrix}
-28546 & 66062 & 39027 & 4998 & -29228 & -83 & -2244 & 51477 & 30411 & 3895 & -22774 & 8 \[34.5] \\
\end{bmatrix}
\]

Decorrelation + ILS with search-and-shrink

\[ [Qz,Zt,Lz,Dz,az,iZ] = \text{decorrel} \ (Q,a) ; \]
\[ [\text{azfixed},\text{sqnorm}] = \text{ssearch} \ (az,Lz,Dz,6) ; \]
\[ \text{afixed} = iZ*\text{azfixed} \]

\[
\begin{bmatrix}
-28546 & 66062 & 39027 & 4998 & -29228 & -83 & -2244 & 51477 & 30411 & 3895 & -22774 & 8 \[34.5] \\
\end{bmatrix}
\]
References


Least-Squares Ambiguity Search technique

This technique requires an approximate solution, which can be obtained from code range measurements. The search area can be defined by surrounding the approximate position by a $3\sigma$ region (i.e. $\Delta \hat{r} \pm \delta$).

\[ y = G \Delta r + \lambda N + v \rightarrow \hat{N} = \frac{1}{\lambda} (y - G \Delta \hat{r}) \]

$K-1$ equations with $3+(K-1)$ unknowns.

Then, given the 3-D position, the $(K-1)$ integer ambiguities can be resolved automatically.

That is, the $(K-1)$ integer ambiguities are constrained to three degrees of freedom.
The technique is based on exploiting the constrains on the integer ambiguities:

\[ \Delta \hat{\mathbf{r}} \pm \sigma \rightarrow \mathbf{N}_{(k)} = \frac{1}{\lambda} \left( \mathbf{y} - \mathbf{G} \left( \Delta \hat{\mathbf{r}} + \delta_{(k)} \right) \right) \rightarrow \mathbf{N}_{(k)} \]

Then, the corresponding position estimates \( \Delta \hat{\mathbf{r}}_{(k)} \) are computed.

\[ \mathbf{N}_{(k)} \rightarrow \Delta \hat{\mathbf{r}}_{(k)} = \left( \mathbf{G}^T \mathbf{G} \right)^{-1} \mathbf{G} \left[ \mathbf{y} - \lambda \mathbf{N}_{(k)} \right] \]

- The tracked satellites into two groups: \( 4 \text{ sat.} \) (with good DOP) + \( N-4 \text{ sat.} \)

- The primary group of \( 4 \text{ sat.} \) is used determine the possible ambiguity sets \( \mathbf{N}_{(k)} \) (given an initial position estimate and its associated uncertainty \( \Delta \hat{\mathbf{r}} \pm \sigma \))

- The remaining \( N-4 \) secondary satellites are used to eliminate candidates of the possible ambiguity sets:
  - Each position estimate is checked against measurements from the secondary group of \( N-4 \text{ satellites} \).
  - With the correct position estimate, the difference between the measured and computed carrier phase should be “close to an integer” for each satellite pair.

\[ \frac{1}{\lambda} \left( \mathbf{y} - \mathbf{G} \Delta \hat{\mathbf{r}}_{(k)} \right) - \mathbf{N}_{(k)} \in \mathbb{Z} ? \]
## List of Acronyms

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<td>AGW</td>
<td>Atmospheric Gravity Waves</td>
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<td>ANTEX</td>
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<tr>
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<td>Satellite-Based Augmentation System</td>
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<tr>
<td>SED</td>
<td>Storm Enhancement Density</td>
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<tr>
<td>SINEX</td>
<td>Solution (Software/technique) INdependent EXchange format</td>
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<tr>
<td>Acronym</td>
<td>Description</td>
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<td>SIS</td>
<td>Signal In Space</td>
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<td>SISRE</td>
<td>Signal In Space Range Error</td>
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<td>SOHO</td>
<td>SOlar Helioscopic Observatory</td>
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<td>SP3</td>
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<td>SPP</td>
<td>Standard Point Positioning</td>
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<td>SPS</td>
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<tr>
<td>SSI</td>
<td>Signal Strength Indicator</td>
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<tr>
<td>STEC</td>
<td>Slant Total Electron Content</td>
</tr>
<tr>
<td>STROP</td>
<td>Slant TROPospheric delay</td>
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<td>Soviet Union</td>
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<td>SV</td>
<td>Space Vehicle</td>
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<td>Time Dilution Of Precision</td>
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<tr>
<td>TEC</td>
<td>Total Electron Content</td>
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<tr>
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<td>TGD</td>
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<tr>
<td>TID</td>
<td>Travelling Ionospheric Disturbance</td>
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<td>Terrestrial Reference Frame</td>
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<td>Conventional Terrestrial Reference System</td>
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<td>United States Naval Observatory</td>
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<tr>
<td>UTC</td>
<td>Coordinated Universal Time</td>
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<tr>
<td>VDOP</td>
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<td>WGS-84</td>
<td>World Geodetic System 84</td>
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<td>ZTD</td>
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